

On the Landscape of Superstring Theory in $D > 10$

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We study a family of unstable heterotic string theories in more than ten dimensions which are connected via tachyon condensation to the ten-dimensional supersymmetric vacuum of heterotic string theory with gauge group $SO(32)$. Calculating the spectrum of these theories, we find evidence for an S-duality which relates type I string theory in ten dimensions with n additional ninebrane-antininebrane pairs to heterotic string theory in $10 + n$ dimensions with gauge group $SO(32 + n)$. The Kaluza-Klein modes of the supercritical dimensions are dual to non-singlet bound states of open strings.

1. Introduction

Superstring theories formulated in dimensions greater than ten have played little role in our understanding of the nonperturbative structure of string theory. In particular, the dualities among string theories have so far connected only a closed set containing the theories in which the superstring propagates in the critical number of spacetime dimensions.

At the perturbative level noncritical string theories are equally consistent [1], [2], [3]. In the background of a dilaton with gradient of appropriate magnitude, strings in supercritical dimensions propagate consistently with the rules of quantum mechanics.

Unlike their critical cousins, supercritical superstring theories lack linearly realized spacetime supersymmetry and as a result they are vulnerable to tadpoles, mass renormalizations, and other perturbative instabilities, but such problems do not fundamentally destroy the consistency of string perturbation theory [4], [5], [6]. The supercritical theories are just as sensible in perturbation theory as the critical ones, and yet we lack a comparable understanding of their nonperturbative behavior.

In this paper we will study two families of supercritical heterotic strings with orthogonal gauge group, which we will call HO^+ and $\text{HO}^{+/-}$. Both are perturbatively unstable against decay to lower dimensions. Tachyon condensation in the HO^+ theory will generically lead to a ten-dimensional supersymmetric string theory. The $\text{HO}^{+/-}$ theory is more unstable. It can also decay to ten dimensions, but $\text{HO}^{+/-}$ theory on a smooth space cannot decay to a supersymmetric theory, or even a stable one. On a space with certain orbifold singularities, $\text{HO}^{+/-}$ can decay to the supersymmetric ten-dimensional HO background; the stabilization relies on the boundary condition of the $\text{HO}^{+/-}$ bulk tachyons at the orbifold singularity.

Examining the chiral fermion spectrum at the singularity, we will find that it matches the chiral fermion spectrum of type I string theory in ten dimensions, with additional $D9-\overline{D9}$ pairs added. We will be led to propose that a suitable compactification of the $\text{HO}^{+/-}$ theory, with a single orbifold singularity of the correct type, plays the role of a strong coupling limit of the unstable type I theory. If correct, this duality provides the first link between the landscape of critical string backgrounds and the supercritical world which lies beyond.

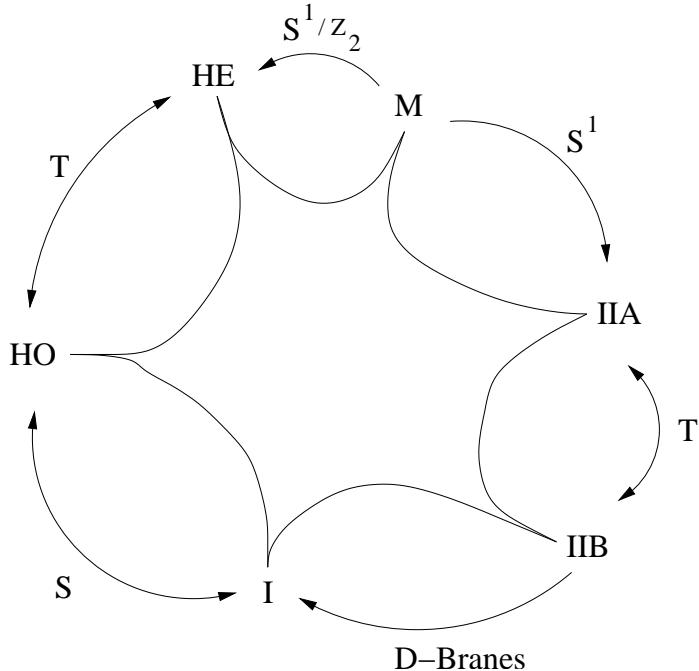


Fig. 1: There are six simple limits of string theory with ten- (or higher-) dimensional super-Poincaré symmetry, connected by known dualities and deformations.

The organization of the paper is as follows. In section 2, we introduce the reader to the basic ideas of superstring theory in more than ten dimensions. In section 3, we present two families of supercritical heterotic string theories, HO^+ and $\text{HO}^{+/-}$, which are connected to the ten-dimensional $SO(32)$ heterotic superstring theory by tachyon condensation. (Attempts to understand the endpoint of closed string tachyon condensation in other string theories include [7], [8], [9], [10].) In sections 3 and 4, we explore local and global aspects of tachyon condensation in HO^+ . In section 5, we consider $\text{HO}^{+/-}$ on orbifolds, with particular attention to the spectrum of chiral fermions at the fixed locus and its correspondence with the fermion spectrum in an unstable type I background. In section 6, we sharpen our S-duality conjecture in certain cases by studying $\text{HO}^{+/-}$ on compact toroidal orbifolds whose ten-dimensional spectrum of chiral fermions matches that of the type I theory with $n = 1, 2$ ninebrane-antininebrane pairs.

2. The basics of supercritical string theory

The basic ideas which allow a consistent interpretation of the supercritical string appear in [1] ; in this section we review them.

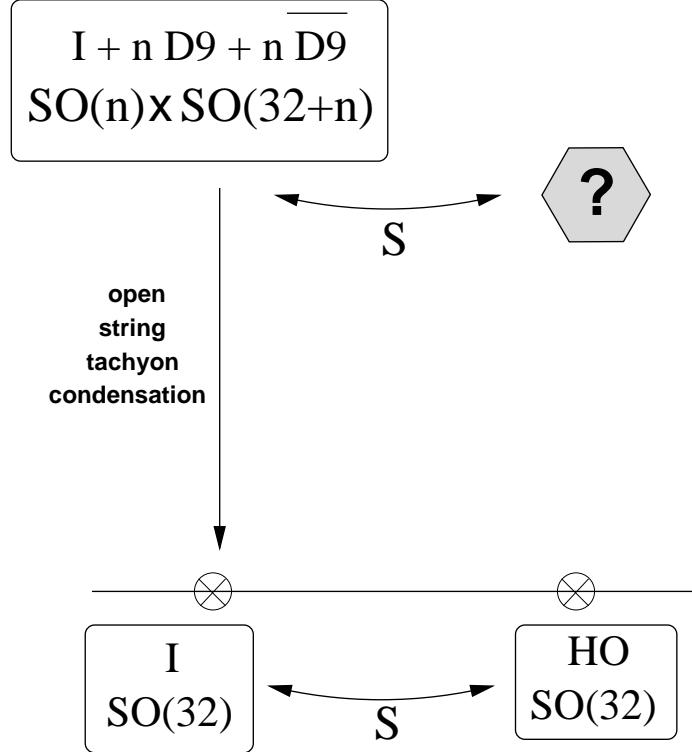


Fig. 2: Nine-dimensionally Poincaré-invariant configurations of string theory which lie above the moduli space. We will fill in the empty spot on the phase diagram with a supercritical heterotic string theory, which is exchanged by S-duality with the type $I+n D9 + n \bar{D9}$ vacuum.

We give particular emphasis to the point that the supercritical string reproduces completely conventional spacetime physics. Despite the presence of a dilaton gradient with string-scale magnitude, spacetime processes are faithfully described by a local effective action ([1], [11], [12], [3], [2]).

2.1. A simple example – maximally Lorentz-invariant type II strings

The physical state conditions and dispersion relations for strings in linear dilaton backgrounds have been analyzed in detail ([1], [3], [2]). For an illustrative example we now analyze the NS sector of the supercritical type II string. The type II string with Lorentz-invariant chiral GSO projection can exist only in dimensions equal to 10 mod 8 ([2], [1]).

The Virasoro generators are

$$\begin{aligned}
L_n^{mat.} - A^{mat.} \delta_{0,n} = & \\
\frac{1}{2} \sum_m : \alpha_{n-m}^\mu \alpha_{\mu m} : + \frac{1}{4} \sum_r (2r-m) : \psi_{n-r}^\mu \psi_{\mu r} : + i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} (n+1) V_\mu \alpha_n^\mu \\
G_r^{mat.} = & \sum_n \alpha_n^\mu \psi_{\mu r-n} + i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} (2r+1) V_\mu \psi_r^\mu \\
& \\
\tilde{L}_n^{mat.} - \tilde{A}^{mat.} \delta_{0,n} = & \\
\frac{1}{2} \sum_m : \tilde{\alpha}_{n-m}^\mu \tilde{\alpha}_{\mu m} : + \frac{1}{4} \sum_r (2r-m) : \tilde{\psi}_{n-r}^\mu \tilde{\psi}_{\mu r} : + i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} (n+1) V_\mu \tilde{\alpha}_n^\mu \\
\tilde{G}_r^{mat.} = & \sum_n \tilde{\alpha}_n^\mu \tilde{\psi}_{\mu r-n} + i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} (2r+1) V_\mu \tilde{\psi}_r^\mu
\end{aligned} \tag{2.1}$$

As always, superghosts contribute -1 to the fermion number of the ground state; so the lowest mode surviving the GSO projection in the NS/NS sector is

$$|\Psi(e, k)\rangle \equiv e_{\mu\nu} \tilde{\psi}_{-\frac{1}{2}}^\mu \psi_{-\frac{1}{2}}^\nu |k; 0\rangle_{\text{NS/NS}} \tag{2.2}$$

We examine the dispersion relations, physical state conditions, and gauge equivalences of this level. Let us consider the simplest case, where $e_{\mu\nu}$ is antisymmetric in μ and ν . The gauge invariance

$$|\Psi\rangle \rightarrow |\Psi\rangle + \tilde{G}_{-\frac{1}{2}}^{mat.} \left| \mathcal{G}^{(-\frac{1}{2}, 0)} \right\rangle + G_{-\frac{1}{2}}^{mat.} \left| \mathcal{G}^{(0, -\frac{1}{2})} \right\rangle \tag{2.3}$$

acts on the state as

$$e_{\mu\nu} \rightarrow e_{\mu\nu} + k_\mu \Lambda_\nu - k_\nu \Lambda_\mu, \tag{2.4}$$

which is the Fourier-space version of the usual gauge transformation

$$B_{\mu\nu}(x) \rightarrow B_{\mu\nu}(x) + \Lambda_{\nu,\mu}(x) - \Lambda_{\mu,\nu}(x). \tag{2.5}$$

We have used the closed-string identification

$$\alpha_0^\mu = \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} k^\mu. \tag{2.6}$$

The gauge parameters $|\mathcal{G}^{(-\frac{1}{2},0)}\rangle$ and $|\mathcal{G}^{(0,-\frac{1}{2})}\rangle$ can be written as

$$|\mathcal{G}^{(-\frac{1}{2},0)}\rangle = (l_\mu + \Lambda_\mu) \psi_{-\frac{1}{2}}^\mu |k, 0\rangle_{\text{NS/NS}} \quad (2.7)$$

and

$$|\mathcal{G}^{(0,-\frac{1}{2})}\rangle = (l_\mu - \Lambda_\mu) \tilde{\psi}_{-\frac{1}{2}}^\mu |k, 0\rangle_{\text{NS/NS}}, \quad (2.8)$$

where l^μ is the vector field parametrizing an infinitesimal coordinate transformation.

The gauge transformation of the string state $|\Psi(e, k)\rangle$ has no dependence on V_μ . This gives us information about the dilaton dependence of its normalization. The state must be interpreted as a linear fluctuation of the $B_{\mu\nu}$ field itself, rather than the field $\exp\{-\Phi_0\}B_{\mu\nu}$ with canonical kinetic term. If the string state represented the canonical field, the gauge transformation law would be different, with a nontrivial dependence on the background dilaton gradient V_μ coming from the action of l_μ on the background value Φ_0 of the dilaton.

The transversality conditions coming from the $\tilde{G}_{\frac{1}{2}}^{mat.}, G_{\frac{1}{2}}^{mat.}$ physical state conditions, do depend on the background value of the dilaton, and they amount to

$$(k + 2iV)^\mu e_{\mu\nu} = k'^\mu e_{\mu\nu} = 0, \quad (2.9)$$

where we define $k' \equiv k + 2iV$. The dispersion relation, which comes from the conditions

$$\begin{aligned} L_0^{mat.} &= \left\{ \begin{array}{ll} \frac{1}{2}, & (\text{NS}) \\ \frac{5}{8}, & (\text{R}) \end{array} \right\} \\ \tilde{L}_0^{mat.} &= \left\{ \begin{array}{ll} \frac{1}{2}, & (\widetilde{\text{NS}}) \\ \frac{5}{8}, & (\widetilde{\text{R}}) \end{array} \right\} \end{aligned} \quad (2.10)$$

and the fact that

$$\begin{aligned} A^{mat.} &= \left\{ \begin{array}{ll} 0, & (\text{NS}) \\ \frac{D}{16}, & (\text{R}) \end{array} \right\} \\ \tilde{A}^{mat.} &= \left\{ \begin{array}{ll} 0, & (\widetilde{\text{NS}}) \\ \frac{D}{16}, & (\widetilde{\text{R}}) \end{array} \right\} \end{aligned} \quad (2.11)$$

says that

$$k^2 + 2iV \cdot k = k \cdot k' = 0 \quad (2.12)$$

for this state. At first it appears puzzling that the dispersion relation is not Lorentz invariant, until we recall ([1], [3], [2]) that the B field appears in the string action as

$$\mathcal{L}_B = \frac{1}{2\kappa_D^2} \sqrt{|G|} \exp\{-2\Phi\} \left[-\frac{1}{12} H_{\mu\nu\sigma} H^{\mu\nu\sigma} \right], \quad (2.13)$$

and that the dilaton Φ has a background value

$$\Phi_0 = \text{const.} + V_\mu X^\mu \quad (2.14)$$

If we linearize the equations of motion about this background, we recover the B-field EOM

$$(\partial^\sigma + 2V^\sigma)(\partial_\sigma B_{\mu\nu} + (\text{cyclic})) = 0. \quad (2.15)$$

Together with the real-space transversality condition

$$(\partial^\mu + 2V^\mu)B_{\mu\nu} = 0 \quad (2.16)$$

which we derived above from the L_1 constraint, the EOM for $B_{\mu\nu}$ is the same as the dispersion relation we obtained from the L_0 (and \tilde{L}_0) condition. A similar analysis goes through for the other NS/NS ground states [1].

3. The landscape of supercritical heterotic string theory

3.1. Two heterotic string theories in eleven dimensions

There are two heterotic string theories with manifest eleven-dimensional Poincaré invariance which will be relevant for us.

The worldsheet degrees of freedom are eleven free embedding coordinates X^μ and their superpartners ψ_+^μ under the right-moving supercharge Q_+ , as well as thirty two left-moving fermions $\tilde{\lambda}_-^a$, with $a = 1, \dots, 32$ and a thirty-third left-moving fermion $\tilde{\chi}$.

We call this theory $\text{HO}^{+(1)}$. The plus stands for the fact that this theory is like the critical HO theory, only with more degrees of freedom on the worldsheet. The 1 stands for the fact that there is a single extra dimension.

The $\text{HO}^{+(1)}$ theory has symmetry group $\text{SO}(32) \times \text{O}(10, 1)$. (The second factor is spontaneously broken by the dilaton gradient but by nothing else.) Unlike the ten-dimensional HO theory, the $\text{HO}^{+(1)}$ theory is invariant under orientation-reversing orthogonal transformations of space.

If there are n extra dimensions, we will refer to the theory as type $\text{HO}^{+(n)}$. When we add n extra embedding coordinates, they will have n right-moving fermionic superpartners. The number of extra left-moving fermions $\tilde{\chi}^A$ must be n in order to cancel the local gravitational anomaly on the worldsheet, by making the left-moving central charge and right-moving central charge equal. The left-moving fermions $\tilde{\lambda}^a, \tilde{\chi}^A$ do not have dynamical superpartners, so their contribution to the left-moving central charge is $\frac{1}{2}$ per extra dimension.

For n extra dimensions, the extra central charge for the right-movers is

$$\Delta c = \Delta c_X + \Delta c_\psi = n + \frac{n}{2} = \frac{3n}{2}. \quad (3.1)$$

The extra central charge for the left-movers is

$$\Delta \tilde{c} = \Delta \tilde{c}_X + \tilde{c}_{\tilde{\chi}} = n + \frac{n}{2} = \frac{3n}{2}. \quad (3.2)$$

The symmetry group of $\text{HO}^{+(n)}$ is $SO(32) \times [O(n) \times O(9+n, 1)]_+$. The second factor denotes the subgroup $(g, g') \in O(n) \times O(9+n, 1)$ with $\det[g] \cdot \det[g'] = 1$.

There are two discrete gauge symmetries on the type HO^+ worldsheet. The first symmetry, g_1 , reverses the sign of all 32 of the $\tilde{\lambda}^a$, just like the $(-1)^{F_L}$ symmetry of type HO. The second, g_2 , reverses the sign of the right-moving fermions ψ_+^μ , as well as the extra left-moving fermions $\tilde{\chi}^A$. g_2 plays the role of $(-1)^{F_{RW}}$ in the supersymmetric HO theory. In particular it is a discrete R-symmetry; the supercharge Q_+ is odd under it.

The second eleven-dimensional heterotic string theory we will discuss has the same field content as HO^+ , and differs in its discrete worldsheet gauge group. In the second theory, we make only one projection on fermions, retaining sectors of even overall fermion number.

We call this theory type $\text{HO}^{+/-}$. The diagonal slash refers to the fact that this theory is the same as HO^+ , only we take the projection on fermions which corresponds to the diagonal modular invariant.

The theory generalizes in the obvious way to the case of n extra dimensions and n extra current algebra fermions. In $10+n$ dimensions, the symmetry group of the $\text{HO}^{+/-}$ theory is $[O(32+n) \times O(9+n, 1)]_+$.

3.2. Spectrum of $\text{HO}^{+(1)}$

The discrete charges of worldsheet fields in this model are as follows:

Table 1: Discrete worldsheet gauge charge assignments in model $\text{HO}^{+(1)}$.

object	$g_1 \simeq (-1)^{F_{LW}}$	$g_2 \simeq (-1)^{F_{RW}}$
Q_+	+	-
X^μ	+	+
ψ_+^μ	+	-
$\tilde{\lambda}_-^a$	-	+
$\tilde{\chi}_-$	+	-

The \simeq above indicates that $g_{1,2}$ act on $\tilde{\chi}_-$ with the opposite of the sign with which they act on all the other left-moving fields.

In order to calculate the spectrum, first we write the super-Virasoro generators for the matter theory in terms of oscillators.

$$\begin{aligned}
& L_n^{mat.} - A^{mat.} \delta_{n,0} = \\
& \frac{1}{2} \sum_m : \alpha_{n-m}^\mu \alpha_{\mu m} : + \frac{1}{4} \sum_r (2r - m) : \psi_{n-r}^\mu \psi_{\mu r} : + i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} (n+1) V_\mu \alpha_n^\mu \\
& G_r^{mat.} = \sum_n \alpha_n^\mu \psi_{\mu r-n} + i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} (2r+1) V_\mu \psi_r^\mu
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
& \tilde{L}_n^{mat.} - \tilde{A}^{mat.} \delta_{n,0} = \\
& \frac{1}{2} \sum_m : \tilde{\alpha}_{n-m}^\mu \tilde{\alpha}_{\mu m} : + \frac{1}{4} \sum_r (2r - m) : \tilde{\lambda}_{n-r}^a \tilde{\lambda}_r^a : + \\
& + \frac{1}{4} \sum_r (2r - m) : \tilde{\chi}_{n-r} \tilde{\chi}_r : + i \left(\frac{\alpha'}{2} \right)^{\frac{1}{2}} (n+1) V_\mu \alpha_n^\mu
\end{aligned}$$

The linear dilaton term changes the central charge by $\Delta c = 6\alpha' V_\mu V^\mu$, so in order to make the total central charge in the matter sector equal to (26, 15), we must take

$$V_\mu V^\mu = -\frac{1}{4\alpha'} (D - 10) \tag{3.4}$$

According the standard rules, explained in ([13], [14]), we compute the normal-ordering contributions $A^{mat.}$ and $\tilde{A}^{mat.}$ to $\tilde{L}_0^{mat.}$ and $L_0^{mat.}$ in each sector. We also list the normal-ordering contributions $\tilde{A}^{gh.}$, $A^{gh.}$ to $\tilde{L}_0^{gh.}$, $L_0^{gh.}$ from the ghosts. As in the usual heterotic string, $\tilde{A}^{gh.}$ is always -1 and $A^{gh.}$ depends only on the periodicity of the supercurrent in each sector.

Table 2: Normal ordering contributions to \tilde{L}_0 and L_0 in various sectors, for the $HO^{+(1)}$ theory.

field ↓	sector →	1	g_1	g_2	$g_1 g_2$
$\tilde{b}, \tilde{c}, b, c, \beta, \gamma$		$(-1, -\frac{1}{2})$	$(-1, -\frac{1}{2})$	$(-1, -\frac{5}{8})$	$(-1, -\frac{5}{8})$
X^μ		$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
ψ_+^μ		$(0, 0)$	$(0, 0)$	$(0, +\frac{11}{16})$	$(0, +\frac{11}{16})$
$\tilde{\lambda}_-^a$		$(0, 0)$	$(+2, 0)$	$(0, 0)$	$(+2, 0)$
$\tilde{\chi}_-$		$(0, 0)$	$(0, 0)$	$(+\frac{1}{16}, 0)$	$(+\frac{1}{16}, 0)$
$(\tilde{A}^{total}, A^{total})$		$(-1, -\frac{1}{2})$	$(+1, -\frac{1}{2})$	$(-\frac{15}{16}, +\frac{1}{16})$	$(+\frac{17}{16}, +\frac{1}{16})$

First we examine the lowest level with momentum k in the untwisted sector $| \rangle_1$. We need to act on the vacuum $|k, 0\rangle_1$ with some object which is odd under the symmetries g_2 and $g_1 g_2$, since the effect of the superghosts is to make the vacuum odd under these elements. Since $\tilde{\chi}$ is odd under both and antiperiodic in the untwisted sector, we can make the state

$$|T(k)\rangle \equiv \tilde{\chi}_{-\frac{1}{2}} |k, 0\rangle_1 \quad (3.5)$$

which satisfies the physical state conditions if and only if

$$k^2 + 2i(V \cdot k) = \frac{2}{\alpha'} \quad (3.6)$$

Changing the k 's into $-i\partial$'s, this gives the linearized EOM for the tachyon

$$\partial^2 T - 2(\partial\Phi_0)(\partial T) + \frac{2}{\alpha'} T = 0, \quad (3.7)$$

which comes from the Lagrangian

$$\frac{1}{\kappa_{11}^2} \exp\{-2\Phi\} \sqrt{|G|} \left[-\frac{1}{2}(\nabla_\mu T)(\nabla^\mu T) + \frac{1}{\alpha'} T^2 \right] \quad (3.8)$$

By a calculation which precisely follows those of ([3], [2], [1]), the rest of the states of the untwisted sector also appear in the action with their usual kinetic terms, in addition to the potential for the dilaton:

$$\mathcal{L}_{\text{untwisted}}^{\text{HO}^+} = \frac{1}{2\kappa_{11}^2} \exp\{-2\Phi\} \sqrt{|G|} \cdot \begin{bmatrix} -\frac{1}{\alpha'} + 4(\nabla_\mu \Phi)(\nabla^\mu \Phi) \\ -\frac{1}{2}(\nabla_\mu T)(\nabla^\mu T) + \frac{1}{\alpha'} T^2 \\ + \mathbf{R} - \frac{1}{12} \tilde{H}_{\mu\nu\tilde{\lambda}} \tilde{H}^{\mu\nu\tilde{\lambda}} \\ - \frac{\kappa_{11}^2}{2g_{11}^2} \text{Tr}_v(F_{\mu\nu}F^{\mu\nu}) + O(\alpha') \end{bmatrix} \quad (3.9)$$

The subscript 'v' denotes a trace taken in the vector representation of $SO(32)$ and the tilded field strength \tilde{H} is given by the sum

$$\tilde{H} \equiv dB - \frac{\kappa_{11}^2}{g_{11}^2} \omega_3, \quad (3.10)$$

of the curl of the B-field and a multiple of the Chern-Simons form

$$\omega_3 \equiv \text{Tr}_v \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (3.11)$$

Considering a certain class of tree-level solutions will serve to fix $\frac{\kappa_{11}^2}{g_{11}^2}$ in terms of α' , but we defer that analysis until later in the section. For now we continue to explore the spectrum of the $\text{HO}^{+(1)}$ theory.

The sectors twisted by g_1 and $g_1 \cdot g_2$ are less interesting for us, containing only massive bosons and fermions, in the sense that we will always use the words 'massive' and 'massless' in this paper: their equations of motion are those of fields with an explicit mass term in the Lagrangian; we will not include such fields in any of the effective actions we consider.

Remaining is the sector g_2 , which contains massless fermions.

Let $|k, \alpha, 0\rangle_{g_2}$ be the twisted vacua, which transform in a representation (with basis labeled by α) of the Clifford algebra

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \quad \{\psi_0^\mu, \tilde{\chi}_0\} = 0 \quad \{\tilde{\chi}_0, \tilde{\chi}_0\} = 1 \quad (3.12)$$

generated by the zero modes of $\tilde{\chi}$ and ψ^μ .

The vacua are not level-matched and cannot be physical states. Nonetheless let us analyze their transformation properties under the symmetries of the theory, since the

physical states which we build by acting with positive-frequency oscillators will inherit gauge and Lorentz quantum numbers from the ground states.

The Clifford algebra is generated by twelve elements whose total signature is $(11, 1)$. The states transform as spinors of $SO(11, 1)$, though the dynamics do not respect this symmetry since one of the eleven positive-signature generators is the zero mode of a left-moving, rather than right-moving fermion.

The properties of spinor representations of this algebra are analyzed in an appendix. The result is that, once the GSO projection and the reality condition are taken into account, is that the ground states transform as a single Majorana spinor of $SO(10, 1)$.

The ground state has $L_0 - \tilde{L}_0 = 1$, so in order to make a level-matched state we must act on the ground state $|\alpha, k, 0\rangle_{g_2}$ with a set of left-moving creation operators of total energy $+1$. So our candidate physical states are

$$\begin{aligned} |\psi_\alpha^{[ab]}(k)\rangle &\equiv \tilde{\lambda}_{-\frac{1}{2}}^a \tilde{\lambda}_{-\frac{1}{2}}^b |\alpha, k, 0\rangle_{g_2}, \\ |\Psi_\alpha^\mu(k)\rangle &\equiv \tilde{\alpha}_{-1}^\mu |\alpha, k, 0\rangle_{g_2}, \quad \text{and} \\ |\theta_{\dot{\alpha}}(k)\rangle &\equiv \tilde{\chi}_{-1} |\dot{\alpha}, k, 0\rangle_{g_2} \end{aligned} \quad (3.13)$$

Since g_2 is a Ramond sector, the on-shell condition for all three comes from the G_0 constraint; physical states satisfy the modified massless Dirac equation

$$(k + iV)_\mu \Gamma_{\alpha\beta}^\mu |\psi_\beta^{[ab]}(k)\rangle = (k + iV)_\mu \Gamma_{\alpha\beta}^\mu |\Psi_\beta^\nu(k)\rangle = (k + iV)_\mu \Gamma_{\dot{\alpha}\dot{\beta}}^\mu |\theta_{\dot{\alpha}}(k)\rangle = 0. \quad (3.14)$$

By 'NS sector' ('Ramond sector') we will mean a sector with boundary conditions which make the supercurrent even (odd).

Notice that the dilaton gradient appears in the combination $k + iV$ in these dispersion relations, rather than the combination $k + 2iV$ with which it appears in the dispersion relations for NS sector fields.

These equations of motion can be obtained from the lagrangian

$$\mathcal{L}_{fermi}^{\text{HO}^+} = \frac{i}{2\kappa_{11}^2} \exp\{-2\Phi\} \sqrt{|G|} [\text{Tr}_v (\bar{\psi} \Gamma^\mu \nabla_\mu \psi) + \bar{\theta} \Gamma^\mu \nabla_\mu \theta + \bar{\Psi}^\nu \Gamma^\mu \nabla_\mu \Psi_\nu] \quad (3.15)$$

Despite the appearance of $k + iV$ in the dispersion relations, there is still the same $\exp\{-2\Phi\}$ multiplying the kinetic term for the fermions as for the massless bosons. The combination $k + iV$ arises because there is only a single derivative in the fermi kinetic action; in the Euler-Lagrange equations, there are two terms in which the derivative acts on the fermions and only one term in which it acts on the dilaton.

The vector-spinor Ψ_α^μ can be decomposed into a spin $1/2$ field and a spin- $3/2$ field transforming according to a fermionic gauge invariance.

3.3. Spectrum of the $\text{HO}^{+(1)}/$ theory

The worldsheet theory of $\text{HO}^{+(1)}/$ has only one discrete gauge symmetry – a projection onto even overall fermion number. The left-moving fermions $\tilde{\lambda}^a$ have an index which runs from one to $n + 32$ and the X^μ, ψ^μ have an index which runs from 0 to $n + 9$.

Table 3: Discrete worldsheet gauge charge assignments in the 11D nonsusy heterotic theory.

object	$g_1 \equiv (-1)^{F_W}$
Q_+	-
X^μ	+
ψ_+^μ	-
$\tilde{\lambda}_-^a$	-

This choice is modular invariant in any dimension, as long the number of current algebra fermions minus the number of spacetime dimensions is equal to 22. The continuous part of the Lorentz group is $SO(n+9, 1)$, spontaneously broken only by the dilaton gradient; the continuous internal gauge group is $SO(n + 32)$.

Specializing to the case of $\text{HO}^{+(1)}/$, we calculate the normal ordering contributions to the theory in its two sectors:

Table 4: Normal ordering contributions to \tilde{L}_0 and L_0 in various sectors, for the $\text{HO}^{+(1)}/$ theory.

field ↓	sector →	1	g_1
$\tilde{b}, \tilde{c}, b, c, \beta, \gamma$		$(-1, -\frac{1}{2})$	$(-1, -\frac{5}{8})$
X^μ		$(0, 0)$	$(0, 0)$
ψ_+^μ		$(0, 0)$	$(0, +\frac{11}{16})$
$\tilde{\lambda}_-^a$		$(0, 0)$	$(+\frac{33}{16}, 0)$
$(\tilde{A}^{total}, A^{total})$		$(-1, -\frac{1}{2})$	$(+\frac{17}{16}, +\frac{1}{16})$

The analysis of the spectrum goes as in the $\text{HO}^{+(1)}$ theory, with three differences:

- The massless gauge bosons $A_\mu^{[ab]}$ obey the same equation of motion and transversality condition as in $\text{HO}^{+(1)}$, but there are more of them since the continuous gauge group is $SO(33)$.

- The tachyons T^a , obtained by acting with $\tilde{\lambda}_{-\frac{1}{2}}^a$ on the untwisted vacuum, are more numerous and transform in the vector representation of $SO(33)$.

- There are no massless fermions at all.

The action generating the free equations of motion for the massless fields looks exactly like the action for the untwisted sector of HO^+ , except that the tachyon is a vector instead of a singlet:

$$\mathcal{L}^{HO^{+/-}} = \frac{1}{2\kappa_{11}^2} \exp\{-2\Phi\} \sqrt{|G|} \cdot \begin{bmatrix} -\frac{1}{\alpha'} + 4(\nabla_\mu \Phi)(\nabla^\mu \Phi) \\ -\frac{1}{2}(\nabla_\mu T^a)(\nabla^\mu T^a) + \frac{1}{\alpha'} T^a T^a \\ + \mathbf{R} - \frac{1}{12} \tilde{H}_{\mu\nu\tilde{\lambda}} \tilde{H}^{\mu\nu\tilde{\lambda}} \\ - \frac{\kappa_{11}^2}{2g_{11}^2} \text{Tr}_v(F_{\mu\nu} F^{\mu\nu}) + O(\alpha') \end{bmatrix} \quad (3.16)$$

The absence of massless spacetime fermions makes the $HO^{+/-}$ theory seem out of place in our discussion. If our interest is in theories which can reach supersymmetric critical vacua by tachyon condensation, what is $HO^{+/-}$ doing here? If the theory could relax to a supersymmetric vacuum, where could the gravitini, dilatini, and gluini possibly come from?

The premise of that question is correct: the $HO^{+/-}$, in its original noncompact version or compactified on any smooth space, can never reach the supersymmetric HO theory by tachyon condensation. But we will show that the $HO^{+(n)/}$ theory on spaces with certain orbifold singularities of codimension n can reach the supersymmetric HO theory by tachyon condensation. These spaces also have chiral fermions living on the singularity.

3.4. Generalization to arbitrary n

At higher n , the HO^+ theory has $10+n$ spacetime embedding coordinates X^μ , thirty-two current algebra fermions $\tilde{\lambda}^a$ and another n current algebra fermions $\tilde{\chi}^A$. The continuous symmetry group is $SO(9+n, 1) \times SO(n) \times SO(32)$, where the first factor rotates the X^μ, ψ^μ coordinates, the second factor rotates the $\tilde{\chi}^A$'s, and the third rotates the $\tilde{\lambda}^a$'s. The $SO(9, 1)$ intercommutes in the usual way with translations to make the ten-dimensional Poincaré group, which is broken down spontaneously to the nine-dimensional Poincaré group by the background dilaton gradient. The discrete gauge charges of the worldsheet theory are given by a table that looks essentially the same as in the case $n = 1$; the only difference is that there are more $\tilde{\chi}^A, X^\mu$, and ψ^μ .

Table 5: Discrete worldsheet gauge charge assignments in model $\text{HO}^{+(n)}$.

object	$g_1 \simeq (-1)^{F_{LW}}$	$g_2 \simeq (-1)^{F_{RW}}$
Q_+	+	-
X^μ	+	+
ψ_+^μ	+	-
$\tilde{\lambda}_-^a$	-	+
$\tilde{\chi}_-^A$	+	-

The table of normal-ordering contributions to the Virasoro generators \tilde{L}_0 and L_0 also looks similar to the table for $\text{HO}^{+(1)}$, though here there is an explicit n -dependence.

Table 6: Normal ordering contributions to \tilde{L}_0 and L_0 in various sectors, for the $\text{HO}^{+(n)}$ theory.

sector → field ↓	1	g_1	g_2	$g_1 g_2$
$\tilde{b}, \tilde{c}, b, c, \beta, \gamma$	$(-1, -\frac{1}{2})$	$(-1, -\frac{1}{2})$	$(-1, -\frac{5}{8})$	$(-1, -\frac{5}{8})$
X^μ	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
ψ_+^μ	$(0, 0)$	$(0, 0)$	$(0, \frac{10+n}{16})$	$(0, \frac{10+n}{16})$
$\tilde{\lambda}_-^a$	$(0, 0)$	$(+2, 0)$	$(0, 0)$	$(+2, 0)$
$\tilde{\chi}_-$	$(0, 0)$	$(0, 0)$	$(\frac{n}{16}, 0)$	$(\frac{n}{16}, 0)$
$(\tilde{A}^{total}, A^{total})$	$(-1, -\frac{1}{2})$	$(+1, -\frac{1}{2})$	$(\frac{n-16}{16}, \frac{n}{16})$	$(\frac{n+16}{16}, \frac{n}{16})$

When we examine the physical state conditions, we use the fact that $V_\mu V^\mu = -\frac{D-10}{4\alpha'}$ and find that the massless NS spectrum consists of a graviton $G_{\mu\nu} - \eta_{\mu\nu}$, a B-field $B_{\mu\nu}$, a dilaton Φ , and a tachyon T^A in the vector representation of $SO(n)$. The gauge transformations of $G_{\mu\nu}$ and $B_{\mu\nu}$ are the usual (dilaton-independent) ones, and the equations of motion are the same as in $\text{HO}^{+(1)}$ when given in terms of the dilaton gradient. For example the B -field obeys the equation of motion

$$(\partial^\sigma - 2V^\sigma)H_{\mu\nu\sigma} = 0. \quad (3.17)$$

As for the Ramond states, the analysis likewise runs in parallel. The new feature comes from the fact that the Ramond ground states, and hence all the massless states in

the g_2 -twisted sector, transform under $SO(n)$ in spinorial representations, because the $\tilde{\chi}^A$ have zero modes in this sector. We derive the content of the Ramond ground states in the appendix; the details of the derivation depend on the value of n mod eight.

Table 7: Normal ordering contributions to \tilde{L}_0 and L_0 in NS and R sectors, for the $\text{HO}^{+(n)}/$ theory.

field ↓	sector →	
	1	$g_1 = (-1)^{F_W}$
$\tilde{b}, \tilde{c}, b, c, \beta, \gamma$	$(-1, -\frac{1}{2})$	$(-1, -\frac{5}{8})$
X^μ	$(0, 0)$	$(0, 0)$
ψ_+^μ	$(0, 0)$	$(0, \frac{10+n}{16})$
$\tilde{\lambda}_-^a$	$(0, 0)$	$(\frac{32+n}{16}, 0)$
$(\tilde{A}^{total}, A^{total})$	$(-1, -\frac{1}{2})$	$(\frac{16+n}{16}, \frac{n}{16})$

3.5. Smooth compactifications of HO^+

We can put these theories on smooth spaces with a curved metric and nonabelian gauge field strengths.

For untwisted perturbations, the theories HO^+ and $\text{HO}^{+}/$ obey the same one-loop β -function equations as one another. Those in turn are the same equations as in the critical heterotic string, except for the term depending on the central charge:

$$\begin{aligned}
 \beta^{G_{\mu\nu}} &= \alpha' \mathbf{R}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\rho\sigma} H_\nu^{\rho\sigma} - \frac{\alpha' \kappa_{11}^2}{g_{11}^2} \text{tr}_v (F_{\mu\sigma} F_\nu^\sigma) + O(\alpha'^2) \\
 \beta^{B_{\mu\nu}} &= -\frac{\alpha'}{2} \nabla^\sigma H_{\mu\nu\sigma} + \alpha' (\nabla^\sigma \Phi) H_{\mu\nu\sigma} + O(\alpha'^2) \\
 \beta^\Phi &= \frac{D-10}{4} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' (\nabla \Phi)^2 - \frac{\alpha'}{24} H^2 - \frac{\alpha' \kappa_{11}^2}{4g_{11}^2} \text{tr}_v (F^2) + O(\alpha'^2) \\
 \beta^{A_\mu^{[ab]}} &= \frac{2\alpha' \kappa_{11}^2}{g_{11}^2} \nabla^\nu F_{\mu\nu}^{[ab]} - \frac{4\alpha' \kappa_{11}^2}{g_{11}^2} (\partial^\nu \Phi) F_{\mu\nu}^{[ab]} + O(\alpha'^2)
 \end{aligned} \tag{3.18}$$

We only show the one-loop beta function here, but we can make a statement about exact solutions to the β -function equations.

Suppose $n = \sum_i^m n_i$ and the internal target manifold \mathbf{X}_n is a product

$$\mathbf{X}_n \equiv \mathbf{X}_{n_1}^{(1)} \times \mathbf{X}_{n_2}^{(2)} \times \cdots \times \mathbf{X}_{n_m}^{(m)} \tag{3.19}$$

of m factors. Suppose further that the vector bundle V_n respects the product structure as well. That is, it factorizes into a product of vector bundles $V_{n_i}^{(i)}$ of rank n_i , each over the space $\mathbf{X}_{n_i}^{(i)}$.

If we choose the factors $(\mathbf{X}_{n_i}^{(i)}, V_{n_i}^{(i)})$ such that the worldsheet sigma model on each has an exact CFT limit, then it is clear the sigma model on \mathbf{X} with bundle V must be conformal as well, since the product of CFT's is itself a CFT.

This gives a simple construction of an infinite class of tree-level solutions to super-critical heterotic string theory. Since the rank of the vector bundle V and the dimension of the internal space \mathbf{X} are both equal to the same number n , these can be interpreted as solutions either to HO^+ or to $\text{HO}^{+/-}$.

The existence of this class of exact solutions serves to fix the coefficients of interaction terms for the untwisted massless levels in the effective action (3.9), (3.16). In particular, it determines [14]

$$\frac{\kappa_{11}^2}{g_{11}^2} = \frac{\kappa_{10}^2}{g_{10}^2} = \frac{\alpha'}{4} \quad (3.20)$$

3.6. Euler number and its generalization

Real vector bundles of rank n over orientable n -dimensional manifolds with structure group $SO(n)$ can be classified according to a certain topological invariant called the *Euler number*.

Consider a section \mathcal{T} of the bundle V , given in a local basis by $T^A(X)$, $A = 1, \dots, n$. Generically \mathcal{T} will vanish only at isolated points on \mathbf{X} . Each vanishing point x_0^i for T^A can be assigned a sign $\sigma \equiv \pm 1$ as follows:

$$\sigma|_{x_0^i} \equiv \text{sign} \left[\frac{dT^1 \wedge dT^2 \wedge \dots \wedge dT^n}{dx^1 \wedge dx^2 \wedge \dots \wedge dx^n} \right]_{x^i=x_0^i} = \text{sign} \det \begin{bmatrix} \frac{\partial T^1}{\partial x^1} & \frac{\partial T^1}{\partial x^2} & \dots & \frac{\partial T^1}{\partial x^n} \\ \frac{\partial T^2}{\partial x^1} & \frac{\partial T^2}{\partial x^2} & \dots & \frac{\partial T^2}{\partial x^n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial T^n}{\partial x^1} & \frac{\partial T^n}{\partial x^2} & \dots & \frac{\partial T^n}{\partial x^n} \end{bmatrix}_{x^i=x_0^i} \quad (3.21)$$

The reader can satisfy herself that deformations of the section can only create isolated zeroes in pairs with opposite sign σ . (A good introduction to this and other aspects of the Euler number for physicists can be found in [15].) Since the manifold is orientable and the structure group of the vector bundle is $SO(n)$, the sign σ of a vanishing point is well defined under a change of local coordinates and local basis for V . Summing σ over

vanishing points x_0^i gives a topological invariant of the pair (\mathbf{X}, V) . This quantity is called the *Euler number* of the vector bundle V over \mathbf{X} .

The definition of this quantity can be extended to the case where the structure group of the product $T_{\mathbf{X}} \otimes V$ of the tangent bundle and the bundle V has overall special orthogonal structure group

$$[O(n) \times O(n)]_+, \quad (3.22)$$

but not $SO(n) \times SO(n)$. The symmetry groups of the HO^+ and $HO^{+/-}$ theories are

$$[O(n+9, 1) \times O(n)]_+ \times SO(32) \quad (3.23)$$

and

$$[O(n+9, 1) \times O(n+32)]_+; \quad (3.24)$$

the compactifications we consider have transition functions in the subgroup $[O(n) \times O(n)]_+$, embedded in the obvious way. This means we can consider spaces where the coordinate transition functions between patches of \mathbf{X} are orientation reversing, so long as the change of local basis on V has a compensating sign of -1 in its determinant, between the same two patches.

Thus, the sum

$$\chi[V] \equiv \sum_{x_0^i} \sigma \quad (3.25)$$

over vanishing points of the section is a topological invariant of the pair (\mathbf{X}, V) .

If we ignore the coordinates X^{0-9} and the current algebra fermions $\tilde{\lambda}^{1-32}$, as well as the ghosts and antighosts, we can consider the $(0, 1)$ supersymmetric sigma model on X with vector bundle V in its own right. This theory by itself is still modular invariant and has only one Z_2 gauge symmetry, overall fermion number mod 2.

In this theory, the Euler number is equal to the Witten index of the quantum theory, in the case where \mathbf{X} is compact and (\mathbf{X}, V) is smooth.

To see this, consider the theory at large volume, where it can be studied semiclassically. Then perturb the Lagrangian with a relevant operator of the form

$$\mathcal{L} = Q_+ \cdot (T^A(X) \tilde{\chi}_-^A), \quad (3.26)$$

which equals

$$\sum_A (T^A(X))^2 - iT^A{}_{,i}(X) \psi_+^i \tilde{\chi}_-^A + A_i^{[AB]} T^B(X) \psi_+^i \tilde{\chi}_-^A \quad (3.27)$$

The index A ranges from 1 to $n \equiv \dim(\mathbf{X})$, so the zeroes of the positive definite worldsheet potential $T^A T^A$ will be isolated for generic perturbations of this form. At an isolated zero, the mass matrix $T^A{}_{,i}$ is nondegenerate.

What is the fermion number (or the fermion number mod 2) of such a vacuum? Changing the sign of the frequency of a fermionic oscillator changes the ground state of that oscillator from $|\uparrow\rangle$ to $|\downarrow\rangle$, and so changes the fermion number of the vacuum by one. This means that $(-1)^F$ for a given vacuum, semiclassically localized at a zero of the section T^A , is given by $\text{sign} \det\{T^A{}_{,i}\}$ evaluated at the point where the section vanishes. It follows that the Witten index is given by summing the quantity $\text{sign} \det\{T^A{}_{,i}\}$ over all zeroes of the tachyon, so the index is equal to the Euler number of (\mathbf{X}, V) . We call this the *generalized Euler number* of the vector bundle V over \mathbf{X} .

In the next section we will see that sections of V can be thought of as configurations of the tachyon, and the generalized Euler number will be the number of disconnected ten-dimensional universes into which the $10+n$ -dimensional universe fragments in the process of tachyon condensation.

4. Tachyon condensation in HO^+ theories

Both HO^+ and $\text{HO}^{+/-}$ have bulk tachyons whose behavior is much like the behavior of the closed string tachyon in the bosonic and type zero theories: a background for the tachyon gives rise to an effective potential on the string worldsheet and destroys the spacetime being probed by the fundamental string.

However the HO^+ and $\text{HO}^{+/-}$ theories differ from the bosonic and type 0 theories in that the latter have gauge-neutral tachyons, whereas in the former all tachyons transform nontrivially under continuous gauge symmetries. We will exploit this property to compactify the theories to ten dimensions in such a way that the ten dimensional effective theories can have stable minima. Indeed we will argue that the minima are exactly the ten-dimensional supersymmetric vacua of the standard HO string theory.

Attempts to understand closed string tachyon condensation in the type 0 and bosonic theories include [8], [10], [16]. In none of these cases is the endpoint of tachyon dynamics a state which is Poincaré-invariant and stable, much less supersymmetric.

We can also perturb the theory by giving an expectation value to the tachyon field. The vertex operator of the tachyon involves a single current algebra fermion mode of frequency $\frac{1}{2}$ acting on the untwisted vacuum, either $\tilde{\chi}_{-\frac{1}{2}}^A$ in the HO^+ theory or $\tilde{\lambda}_{-\frac{1}{2}}^a$ in the $\text{HO}^{+/-}$ theory.

4.1. *Dynamical vs. kinematical tachyon condensation*

We want to show that our theories are connected via tachyon condensation to the supersymmetric vacuum of the ten-dimensional supersymmetric HO theory.

When we say 'connected' by tachyon condensation, we could mean one of two things. We could mean that there exists a dynamical solution

$$G_{\mu\nu}(X), T^A(X), \Phi(X) \quad (4.1)$$

to the equations of motion of string theory which approaches some solution of the supercritical theory – for instance, the timelike linear dilaton background – as $X_0 \rightarrow -\infty$, and to the ten-dimensional HO theory as $X^0 \rightarrow +\infty$. Such an interpretation is appealing and possibly right, but many problems would be involved in making it work, or even in stating precisely what spacetime fields such a solution would be described in terms of.

We will not consider the question of whether such solutions exist. In this paper, when we refer to connecting the supercritical theory to the critical theory by tachyon condensation, we will always mean only that the two theories can be connected by a renormalization group flow on the string worldsheet, where the relevant operator perturbing the supercritical worldsheet is the one which appears in the vertex operator for the corresponding tachyon. We will think of this as a kind of off-shell, or kinematical, tachyon condensation.

4.2. *Local models for tachyon condensation*

Let us now compare and contrast local models of tachyon condensation in various unstable string theories.

Open string tachyons in bosonic string theory

In open bosonic string theory the relevant operator corresponding to open string tachyon condensation is a boundary cosmological constant:

$$\Delta\mathcal{L}_{\text{worldsheet}} = E_{\text{boundary}} \cdot \delta(\partial\Sigma) \quad (4.2)$$

We can give this a dependence on the target space dimensions as long as we keep the scale of the variation large enough so that the contributions to the anomalous dimension is small and the operator is still relevant:

$$\Delta\mathcal{L}_{\text{worldsheet}} = T(X) \cdot \delta(\partial\Sigma), \quad (4.3)$$

where $T(X)$ can be thought of as representing the bosonic open string tachyon. When $T(X) = 0$ the worldsheet theory describes open strings propagating freely in all of space-time; there is a single space-filling D-brane. Turning on the tachyon $T(X) = \text{const.} > 0$ creates a potential energy which suppresses boundaries in the path integral and leaves behind a theory of closed strings only. Turning on $T(X) = \text{const} < 0$ creates a potential which *favors* boundaries; for this choice the string is unstable to nucleation of endpoints and the path integral is not bounded.

One can also turn on the tachyon inhomogeneously in space. The profile $T(X) = m^2 X^2$ creates a potential which in the infrared confines the string endpoints to $X = 0$. The tachyon profile is varying in what can be thought of locally as a kind of kink solution. It can be extended to a true 'kink' profile globally by letting

$$T_{\text{kink}}(X) = 1 - \frac{1}{\cosh \frac{X}{L}}, \quad (4.4)$$

with L some distance scale much larger than α' . But the kink is nontopological; the value of the tachyon $T(X)$ is the same for $X \rightarrow +\infty$ as for $X \rightarrow -\infty$. As a result the kink is unstable; the zero of the potential can be removed without altering the asymptotics of the solution:

$$T_{\text{kink}}(X) \rightarrow 1 - \frac{(1 - T_{D-1})}{\cosh \frac{X}{L}} \quad (4.5)$$

If we give T_{D-1} a small positive value, the profile $T_{\text{kink}}(X)$ no longer has a zero anywhere, and string endpoints are banned from all of space. The notation T_{D-1} is meant to highlight the fact that the parameter T_{D-1} functions as a relevant operator which destroys the codimension-one D-brane defined by the zero of the undeformed kink solution.

More precisely, the perturbation $T_{\text{kink}}(X)$ flows to the theory of a single codimension-one D-brane. That infrared theory has a relevant operator corresponding to the open string tachyon on the D-brane. That relevant operator can be lifted to the ultraviolet, and when lifted it becomes the parameter T_{D-1} .

Open string tachyons in type IIB string theory

One way to represent the theory of a space-filling brane-antibrane pair is to let a free complex fermion $\gamma \neq \gamma^\dagger$ live on each boundary of the worldsheet [17], [18]. Then the two states $|\uparrow\rangle$ and $|\downarrow\rangle$ for the fermion γ at each boundary describe two possible states of each endpoint. The state $|\uparrow\rangle$ can be interpreted as an endpoint living on the brane, and the state $|\downarrow\rangle$ can be interpreted as the endpoint living on the antibrane. Then the ground state fermion number mod two of the brane-antibrane strings and the fermion number

mod two of the brane-brane strings in the matter sector are what they should be: odd and even, respectively. This is why the brane-antibranes strings are tachyonic: their oscillator ground state is already odd, so they do not need a positive-frequency fermionic oscillator to act on them in order to satisfy the GSO condition.

The vertex operator of the open-string tachyon which annihilates the brane-antibrane pair is made from the fermion γ with appropriate spatial and temporal dependence:

$$\Delta\mathcal{L} = \delta(\partial\Sigma) Q \cdot W, \quad W \equiv T(X)\gamma + h.c., \quad (4.6)$$

where Q is a Hermitean supercharge and γ transforms as a Fermi multiplet:

$$Q \cdot \gamma = F \quad Q \cdot F = \dot{\gamma}. \quad (4.7)$$

After we integrate out the auxiliary field, the bosonic potential is $T(X)^2$. Note that this is positive definite and symmetric under $T \rightarrow -T$. Since the fermion γ is complex, T is also a complex function. We can consider various possible local forms of T . If $T \sim X_1 + o(X^2)$ near the origin, then the brane is codimension one. Eightbranes in type IIB string theory are uncharged and unstable against decay, so there should be an open string tachyon in the CFT to which the theory with this superpotential flows.

The relevant operator corresponding to the open string tachyon can be lifted to the UV theory; it corresponds to the operator $i\gamma - i\gamma^\dagger$. Perturbing the boundary superpotential with this operator, with coefficient ϵ , destroys the brane. The total superpotential becomes

$$W = \gamma(X_1 + i\epsilon) + h.c., \quad (4.8)$$

and the bosonic potential is

$$|X_1 + i\epsilon|^2, \quad (4.9)$$

which vanishes nowhere.

We can also choose the local form of the tachyon profile to be

$$T = X_1 + iX_2 \quad (4.10)$$

This represents a stable, codimension-two brane in the type IIB theory. The bosonic potential is

$$|X_1 + iX_2|^2 \quad (4.11)$$

Any perturbation of the tachyon which is bounded at infinity leaves intact the zero of the boundary potential.

Closed string tachyons in bosonic string theory.

The vertex operator for the closed string tachyon is just the identity, dressed with spatial dependence. A spatially homogeneous tachyon perturbation is a cosmological constant on the worldsheet $\Delta\mathcal{L} = \text{const.} = E$; a spatially varying tachyon is a potential for the X^i fields

$$\Delta\mathcal{L} = T(X). \quad (4.12)$$

Just as we did for the bosonic open string tachyon, we can choose $T(X)$ to depend on a single spatial coordinate X_1 in such a way that it has a single quadratic minimum at zero:

$$T(X) = T(X_1) = \frac{1}{2}m^2X_1^2. \quad (4.13)$$

If we began, for example, with a supercritical bosonic string theory in $26 + n$ dimensions, the IR limit of this theory describes a bosonic string in $26 + (n-1)$ dimensions.¹ Just as in the bosonic theory, we can perturb the tachyon profile by lifting its minimum away from zero:

$$\delta T = \text{const.} = \epsilon; \quad (4.14)$$

in the IR this perturbation becomes the bosonic closed string tachyon in $26 + (n-1)$ dimensions.

Closed string tachyons in type HO⁺⁽¹⁾ string theory

This theory has a single real tachyon T which couples to the worldsheet as

$$\Delta W = \tilde{\chi} T(X). \quad (4.15)$$

The bosonic potential on the worldsheet is

$$T(X)^2. \quad (4.16)$$

Locally near $X^{10} = 0$ we can choose $T(X)$ to be of the form

$$T(X) \sim \frac{m}{\sqrt{2}}X^{10}, \quad (4.17)$$

¹ In order for this to be true, the linear dilaton coefficient must also be renormalized. This can only happen through the dressing of this solution with dependence on the time coordinate X^0 dependence. The effect cannot show up in the kinematical approach we are taking to tachyon condensation.

and the resulting bosonic worldsheet potential is

$$\frac{1}{2}m^2(X^{10})^2 \quad (4.18)$$

This has a single zero, at $X^{10} = 0$. So the endpoint of tachyon condensation is a single universe with ten dimensions. It is easy to see that the ten-dimensional universe has no tachyons which lift to deformations of the eleven-dimensional solution; the persistence of a zero of the potential is stable under small perturbations to the superpotential.

Closed string tachyons in type HO⁺⁽ⁿ⁾

The theory has n real tachyons which couple to the worldsheet as $\Delta W = \tilde{\chi}^A T^A(X)$. We can restrict our attention to profiles which break the $SO(n)$ gauge symmetry and $SO(n)$ rotational symmetry of the internal dimensions down to a diagonal subgroup:

$$T^A = \sum_i \delta^{A+9,i} X^i \cdot f(r), \quad r \equiv \sqrt{X^B X^B}. \quad (4.19)$$

There is a single zero of the tachyon at the origin and this zero persists under arbitrary small perturbations, even those which break the symmetry. At the level of a local model, we can just take f to be a constant, $\frac{m}{\sqrt{2}}$. The resulting worldsheet interaction gives a mass term for X^{10} through X^{9+n} and their superpartners, and also for all the $\tilde{\chi}^A$.

4.3. Descent relations for chiral fermion spectra

We do not know how to describe the dynamical change in the number of dimensions in any kind of effective field theory. In the massless bosonic sector, describing the decrease in the number of spacetime degrees of freedom may require mechanisms not yet known.

For the case of massless fermions, the situation is much easier to understand. Upon condensing a tachyon or set of k tachyons inhomogeneously, in order to go from D dimensional string theory to $D - k$ dimensional string theory, the tachyon defect which represents the lower-dimensional universe can be thought of as an ordinary field theory soliton. The chiral fermion spectrum of the $D - k$ dimensional string theory comes out right if we just treat the computation of the fermion spectrum as we would for the case of fermions localized to an ordinary gauge theory defect.

The only inputs necessary for this calculation are the form of the Yukawa coupling between the higher-dimensional tachyon and fermions, and the topology of the tachyon profile, which we discussed earlier in the section. We focus on the $SO(32)$ adjoint fermions and the case $D = 10 + n$, $k = n$.

n = 1

The coupling of the adjoint fermions $\psi_\alpha^{[ab]}$ to the tachyon in the $\text{HO}^{+(1)}$ theory is of the form

$$\int d^{11}x \ y \ T \ \bar{\psi}^{[ab]} \psi^{[ab]}, \quad (4.20)$$

where y is a Yukawa coupling which can be determined from a three-point amplitude on the sphere. Without the tachyon T , no mass term is possible for the $\psi_\alpha^{[ab]}$ field because parity is an exact symmetry of $\text{HO}^{+(1)}$. Under spacetime reflections across the plane $x^i = 0$, the fermions transform as

$$\psi \rightarrow \Gamma^i \psi. \quad (4.21)$$

The eleven-dimensional gamma matrices are real and symmetric (except for Γ^0 which is real and antisymmetric), and so a mass term would transform under parity as

$$M \bar{\psi}^{[ab]} \psi^{[ab]} \rightarrow -M \bar{\psi}^{[ab]} \psi^{[ab]}. \quad (4.22)$$

The tachyon T is a pseudoscalar, so the Yukawa coupling involving T is allowed.

It is interesting to see what happens to the fermion spectrum when the tachyon is condensed inhomogeneously in a kink solution. First consider the local approximation to the kink, where $T(X) = mX^{10}$. Linearizing about this background, the equation of motion for $\psi_\alpha^{[ab]}$ becomes

$$\Gamma_{\alpha\beta}^\mu \partial_\mu \psi_\beta^{[ab]} + myX^{10}\psi_\alpha = 0 \quad (4.23)$$

Consider a state which is a plane wave in the X^{0-9} directions. In order for it to be massless in the X^M directions, ψ must be a zero mode of the operator $\partial_{10} + myX^{10}\Gamma^{10}$. A zero mode of that operator looks like

$$\psi_\alpha^{[ab]} = \exp\left\{-\frac{1}{2}my\hat{\Gamma}^{10} \cdot (X^{10})^2\right\}_{\alpha\beta} \psi^{(0)[ab]}_\beta, \quad (4.24)$$

which is normalizable only if

$$\Gamma^{[10]}\psi^{(0)[ab]} = +\psi^{(0)[ab]} \quad (4.25)$$

so the massless adjoint fermions of the ten-dimensional critical string theory have definite chirality as 10D spinors.

This result is entirely independent of the global form of the tachyon profile; if $T(X^{10})$ is any profile interpolating between two minima $-T_0$ and $+T_0$ of a generic double-well tachyon potential, a zero mode of the linearized action will have the form

$$\psi_\alpha^{[ab]} = \exp\{-y \int_0^{X^{10}} ds T(s) \hat{\Gamma}^{10}\}_{\alpha\beta} \psi_\beta^{(0)[ab]}. \quad (4.26)$$

The integral in the exponent goes as $T_0|X^{10}|$ for large $|X^{10}|$. So the massless $SO(32)$ adjoint fermions in the ten-dimensional theory are exactly those with the chirality $\Gamma^{10}\psi^{[ab]} = \frac{T_0}{|T_0|}\psi^{[ab]}$.

General $n > 1$

Instead of dealing individually with all eight values of $n \bmod 8$, we will use a trick to treat them all uniformly. We will simply treat the spinors ψ as complex Dirac spinors of $SO(n)$ and $SO(n+9,1)$ for the purposes of solving for the zero modes; the reality and GSO conditions can be imposed afterwards. Since the linear operators multiplying the Γ^μ and γ^A matrices are all real, the zero mode equation commutes with the imposition of a reality condition. And since the Γ^μ and γ^A matrices anticommute with the overall chirality operator $\Gamma\gamma$, the zero mode equation respects the GSO condition as well.

So let ψ be a complex Dirac spinor of $SO(9+n,1)$ and of $SO(n)$. In terms of this object the coupling to the tachyon in $10+n$ dimensions can be written as

$$\int d^{10+n}x i\bar{\psi}\Gamma^\mu\nabla_\mu\psi + yT^A\bar{\psi}\gamma^A\psi \quad (4.27)$$

The linearized equation of motion for ten-dimensional massless fermion modes is

$$i\Gamma^i\partial_i\psi + y\gamma^A T^A\psi = 0, \quad (4.28)$$

where the sum over i is taken to run from 10 to $9+n$. For the local model, we can take $T^A = \frac{m}{\sqrt{2}}x^{A+9}$. In that case, the operators $\mathcal{O}_A \equiv \Gamma^{A+9}\partial_{A+9} - \frac{iy}{\sqrt{2}}x^{A+9}\gamma^A$ anticommute with one another, and annihilate ψ . For $ym > 0$, this means all spinor components are zero except those with eigenvalue +1 under $-i\Gamma^{A+9}\gamma^A$. This cuts down the number of spinor components by a factor of 2^n , exactly the right number to give a single massless spinor in ten dimensions. This applies to all the HO^+ massless fermion states.

4.4. Baby universes from tachyon condensation

In the subsection above, we have seen that sections T^A of the vector bundle V (with sufficiently slow spatial variation) correspond to relevant perturbations of the worldsheet action. These in turn can be thought of as off-shell configurations for the tachyon.

In a compact space this tells us something interesting about the endpoint of tachyon condensation. If χ is the Euler number of the vector bundle V over our compact space \mathbf{X} , then there are at least χ zero energy worldsheet vacua. Since this property is a result of worldsheet supersymmetry it is entirely independent of α' corrections to the 2D couplings. Generically these vacua are isolated from one another, and so we are led to expect that dynamical tachyon condensation will lead to a set of at least $\chi[V]$ disconnected universes.

For pairs (V, \mathbf{X}) with $\chi = 0$, the 10-dimensional tachyons have the same property as the bosonic and type zero tachyons: condensing them in a generic way does not lead to any conformal two-dimensional worldsheet theory. For $\chi = 0$ the effective tachyon potential in 10 dimensions does not have a stable minimum describing a perturbative string theory.

For (V, \mathbf{X}) with $\chi \geq 2$, condensing the tachyon leads to a worldsheet theory with multiple minima, and the IR theory describes strings propagating in several disconnected universes. The consistency of an evolution from a single connected universe to multiple disconnected universes in a quantum theory is unclear. At best, such an evolution would raise difficult questions about unitarity and the flow of information. In addition, it would not seem possible to describe tachyon condensation for $\chi \geq 2$ with any sort of a conventional ten-dimensional effective field theory. The fluctuations about the stable vacuum would have to include more than one ten-dimensional graviton state.

Only for the case $\chi = 1$ is it possible to describe tachyon condensation to a supersymmetric minimum in terms of ten-dimensional effective field theory in a single universe. Therefore we should restrict our attention to pairs (V, \mathbf{X}) of Euler number $\chi = 1$ if we want closed string tachyon condensation to be described by conventional ten-dimensional effective dynamics with a supersymmetric vacuum.

Pairs (V, \mathbf{X}) with $\chi = 1$ are rare. One example is the product

$$\mathbf{X} = (\mathbb{R}\mathbb{P}_2)^k \tag{4.29}$$

of k real projective planes, with V being the tangent bundle to \mathbf{X} . We know of no compact examples at all in which \mathbf{X} is smooth and Ricci-flat. Because of that, we will not pursue tachyon condensation in HO^+ theories beyond the end of this section.

Some caution must be applied when using the intuition with which we have argued for the evolution to disconnected universes when $\chi \geq 2$. Our counting of baby universes relies on an *adiabatic approximation*, in which time evolution is accurately described by a 2D field theory whose couplings evolve with X^0 . Such an approximation may not be valid in an $n + 1$ -dimensional background in which tachyons are condensing dynamically.

For one thing, the characteristic wavelength of the tachyon is $\sqrt{\alpha'}$, and so the metric of an $n + 1$ -dimensional solution will likely have variation on string scale due to the back-reaction of T^A on the metric. As a result, $O(\alpha')$ corrections to the β -function equations cannot be ignored. Though we have argued that $O(\alpha')$ corrections to the sigma model couplings cannot change the minimum number of baby universes at $X^0 \rightarrow \infty$, it is still possible that the α' -corrections will sum up in such a way that the 2D theory undergoes a transition, past which it cannot be described by a sigma model at all.

More surprisingly, the adiabatic approximation can break down by another mechanism even when all curvatures and gradients are arbitrarily small. This is best illustrated by an example. Let \mathbf{X} be a smooth orientable space of dimension $n = 6$ and let V be the tangent bundle of \mathbf{X} . Then let (\mathbf{X}', V') be another such pair with Euler number χ' .

There exists a smooth Euclidean space Y_7 with boundary, such that the oriented boundary ∂Y_7 of Y_7 is equal to the disconnected sum of \mathbf{X} and \mathbf{X}' . This fact follows from the triviality of the *oriented cobordism group* in dimension six. (For an introduction to this set of concepts, see for example [19].)

If we are allowed to think of Y_7 as an 'off-shell' configuration contributing to the Euclidean path integral of string theory, then there is no selection rule or principle of continuity which forbids smooth histories in which pairs (\mathbf{X}, V) change their Euler number. Such histories can have arbitrarily small curvature, so α' -corrections to the effective action do not serve to suppress them.

The oriented cobordism groups are trivial or cyclic in many dimensions other than six [19]. Taking into account the necessity of defining spinors consistently on the interpolating manifold Y does provide some additional selection rules [20], but even these fail to suppress most processes which would violate the Euler number.

5. A duality between critical and supercritical string theories

5.1. $\text{HO}^{+(1)}/$ on spaces with boundary

Smooth compactifications in the $\text{HO}^{+}/$ theory will not be of interest to us.

Instead we would like to study the $\text{HO}^{+}/$ theory on orbifolds. Specifically, we want to study the behavior of $\text{HO}^{+(n)}/$ near fixed loci of Z_2 orbifold actions which reflect n of the spatial coordinates. We will call this orbifold group element g_2 .

Start with the case $n = 1$. We will orbifold $\mathbb{R}^{10,1}$ by reflection of the tenth spatial coordinate $X^{10} \rightarrow -X^{10}$. To preserve $(0,1)$ SUSY we must also act on ψ^{10} as $\psi^{10} \rightarrow -\psi^{10}$.

This action in itself would lead to an orbifold without a modular invariant partition function. The requirement of modular invariance is satisfied for an orbifold which is level-matched in the twisted sector. Since every twisted fermion contributes $+\frac{1}{16}$ to the ground state energy, it is sufficient to require that the number of left-moving fermions odd under each Z_2 differ from the number of right-moving fermions odd under the same Z_2 by a multiple of 16.

For the case $n = 1$ there is exactly one right-moving fermion ψ^{10} which is odd under the orbifold action g_2 , so we can get a modular-invariant theory by acting with a minus sign on all 33 of the left-moving fermions $\tilde{\lambda}^a$ with g_2 . Below we list the transformations of all the objects in the theory under the two new elements of the discrete gauge group.

Table 8: Charges of worldsheet fields under g_1 and g_1g_2 in $\text{HO}^{+(1)}$ on the half-line \mathbb{R}/Z_2 . g_1 is the worldsheet fermion number mod two symmetry $(-1)^{F_W}$ and g_2 is a reflection of the eleventh dimension, along with an inversion of all 33 $\tilde{\lambda}^a$ fermions.

object ↓	orbifold group element \rightarrow	g_2	g_1g_2
Q_+		+	-
X^{0-9}		+	+
X^{10}		-	-
ψ_+^{0-9}		+	-
ψ_+^{10}		-	+
$\tilde{\lambda}_-^a$		-	+

Every element of the orbifold group is of order 2. In the sector twisted by an element h , each free left-moving field odd under h , bose or fermi, contributes $+\frac{1}{16}$ to the ground

state value of \tilde{L}_0 in that sector; the same applies to right-moving fields and their contributions to the ground state value of L_0 . The ghosts and superghosts also make their usual contributions to the ground states of NS and R sectors.

Table 9: Ground state contributions to \tilde{L}_0 and L_0 in twisted sectors of $\text{HO}^{+(1)}/$ on the half-line \mathbb{R}/Z_2 .

object ↓	sector →	g_2	g_1g_2
$\tilde{b}, \tilde{c}, b, c, \beta, \gamma$		$(-1, -\frac{1}{2})$	$(-1, -\frac{5}{8})$
X^{0-9}		$(0, 0)$	$(0, 0)$
X^{10}		$(\frac{1}{16}, \frac{1}{16})$	$(\frac{1}{16}, \frac{1}{16})$
ψ_+^{0-9}		$(0, 0)$	$(0, \frac{5}{8})$
ψ_+^{10}		$(0, \frac{1}{16})$	$(0, 0)$
$\tilde{\lambda}_-^a$		$(\frac{33}{16}, 0)$	$(0, 0)$
$(\tilde{A}^{total}, A^{total})$		$(+\frac{9}{8}, -\frac{3}{8})$	$(-\frac{15}{16}, +\frac{1}{16})$

The sector g_2 will not be of interest to us; it contains only massive fields, due to the large ground state energy contributed by the periodic $\tilde{\lambda}^a$. We will focus on the sector g_1g_2 , which contains states which are massless in the sense of ([1], [2], [3]).

The coordinate X^{10} is odd under g_1g_2 , so the states in this sector are localized to the fixed plane $X^{10} = 0$. The fermions ψ^{0-9} are also odd under g_1g_2 , so the states are spinors of $SO(9, 1)$. The fermion ψ^{10} is untwisted, so these spinor states are *not* representations of the full $SO(10, 1)$, only of the $SO(9, 1)$ which acts on the fixed plane. The supercharge Q_+ is odd under g_1g_2 , so to preserve modular invariance we insert a (-1) into the partition function in this sector; as a result these states are spacetime fermions.

Label the oscillator vacua in this sector by

$$|(k_0, \dots, k_9), \alpha, 0\rangle_{g_1g_2} \quad (5.1)$$

and

$$|(k_0, \dots, k_9), \dot{\alpha}, 0\rangle_{g_1g_2} \quad (5.2)$$

where α and $\dot{\alpha}$ are parametrize bases for positive- and negative-chirality spinors of $SO(9, 1)$, respectively. As explained in the appendix, fermion zero modes ψ_0^M ($M = 0, \dots, 9$) act on the oscillator vacua as

$$\psi_0^M |k^N, \alpha, 0\rangle_{g_1 g_2} = \frac{1}{\sqrt{2}} \sum_{\dot{\alpha}} \Gamma_{\alpha \dot{\alpha}}^M |k^N, \dot{\alpha}, 0\rangle_{g_1 g_2} \quad \text{and} \quad (5.3)$$

$$\psi_0^M |k^N, \dot{\alpha}, 0\rangle_{g_1 g_2} = \frac{1}{\sqrt{2}} \sum_{\alpha} \Gamma_{\dot{\alpha} \alpha}^M |k^N, \alpha, 0\rangle_{g_1 g_2}$$

where the $\Gamma_{\alpha \dot{\alpha}}^M$ and $\Gamma_{\dot{\alpha} \alpha}^M$ are real 16×16 matrices satisfying the Dirac algebra

$$\begin{aligned} \Gamma_{\alpha \dot{\alpha}}^M \Gamma_{\dot{\alpha} \beta}^N + (M \leftrightarrow N) &= 2\delta_{\alpha \beta} \\ \Gamma_{\dot{\alpha} \alpha}^M \Gamma_{\alpha \dot{\beta}}^N + (M \leftrightarrow N) &= 2\delta_{\dot{\alpha} \dot{\beta}} \end{aligned} \quad (5.4)$$

In order to obtain a level-matched state we need to act on the oscillator vacua with a set of raising operators for the left-moving oscillators, of total energy $+1$. There are several ways to do this.

Acting with $\tilde{\alpha}_{-1}^M$ gives us a 10-dimensional Majorana-Weyl vector-spinor field of each chirality:

$$\begin{aligned} \Psi_{\alpha}^M &\leftrightarrow \tilde{\alpha}_{-1}^M |k, \alpha, 0\rangle_{g_1 g_2}, \\ \text{and} \\ \Psi_{\dot{\alpha}}^M &\leftrightarrow \tilde{\alpha}_{-1}^M |k, \dot{\alpha}, 0\rangle_{g_1 g_2}. \end{aligned} \quad (5.5)$$

Acting with two $\tilde{\lambda}_{-\frac{1}{2}}^a$ operators gives us a ten-dimensional Majorana-Weyl spinor in the adjoint of $SO(32)$:

$$\psi_{\alpha}^{[ab]} \leftrightarrow \tilde{\lambda}_{-\frac{1}{2}}^a \tilde{\lambda}_{-\frac{1}{2}}^b |k, \alpha, 0\rangle_{g_1 g_2}, \quad (5.6)$$

and similarly for $\dot{\alpha}$. The oscillators $\tilde{\alpha}^{10}$ are also half-integrally moded in this sector like the $\tilde{\lambda}^a$, since X^{10} is odd under $g_1 g_2$. So we can make another set of states by acting on the oscillator vacua with one $\tilde{\lambda}^a$ raising operator and one $\tilde{\alpha}^{10}$ raising operator:

$$\Upsilon_{\alpha}^a \leftrightarrow \tilde{\lambda}_{-\frac{1}{2}}^a \tilde{\alpha}_{-\frac{1}{2}}^{10} |k, \alpha, 0\rangle_{g_1 g_2}, \quad (5.7)$$

and similarly for $\dot{\alpha}$. We can also make a level-matched state by acting with two α^{10} raising operators:

$$\theta_{\alpha} \leftrightarrow \tilde{\alpha}_{-\frac{1}{2}}^{10} \tilde{\alpha}_{-\frac{1}{2}}^{10} |k, \alpha, 0\rangle_{g_1 g_2}, \quad (5.8)$$

and similarly for $\dot{\alpha}$.

Having added twisted sectors, we now restrict to gauge-invariant states. In the untwisted sector, physical states must be even under the combination of $X^{10} \rightarrow -X^{10}$ and $\tilde{\lambda}^a \rightarrow -\tilde{\lambda}^a$. For scalars this means that states transforming as odd-rank tensors under $SO(33)$ must vanish at the fixed plane $X^{10} = 0$, and states transforming as even-rank tensors must have vanishing normal derivative at the fixed plane. In particular, this means that the tachyon T^a has the Dirichlet boundary condition $T^a = 0$ at $x^{10} = 0$. We shall see that this is quite important, as it will have the effect of stabilizing the system of localized modes at the boundary when the tachyon is condensed; the boundary condition for the tachyon prevents the bulk instability from reaching the boundary. In this sense, the system described in this section is the precise opposite of that described in [7], which describes a stable string theory with a tachyon localized at an orbifold fixed plane.

Tensors get an extra minus sign under g_2 and $g_1 g_2$ for each index along the X^{10} direction. We list the resulting boundary conditions for bulk fields in a table below:

Table 10: Boundary conditions for bulk fields at an \mathbb{R}^n/Z_2 orbifold singularity of stable type in type $HO^{+(1)}/$ string theory.

Bulk field	Boundary condition at $x^{10} = 0$
T^a	Dirichlet
G_{MN}, B_{MN}	Neumann
$G_{M\ 10}, B_{M\ 10}$	Dirichlet
$G_{10\ 10}$	Neumann
Φ	Neumann
$A_M^{[ab]}$	Neumann
$A_{10}^{[ab]}$	Dirichlet

In the $g_1 g_2$ -twisted sector, the GSO projection has the effect of selecting one of the two chiralities of Majorana-Weyl spinor, for each state of the positive-frequency oscillators. Since $\tilde{\alpha}_{-1}^M$, $\tilde{\lambda}_{-\frac{1}{2}}^a \tilde{\lambda}_{-\frac{1}{2}}^b$, and $\tilde{\alpha}_{-\frac{1}{2}}^{10} \tilde{\alpha}_{-\frac{1}{2}}^{10}$ are invariant under the orbifold group, the corresponding spinors all have the same 10D chirality, say left-handed. $\tilde{\alpha}_{-\frac{1}{2}}^{10} \tilde{\lambda}_{-\frac{1}{2}}^a$ is even under g_2 but odd under g_1 , so it has the same quantum numbers as ψ^M . This means the state obtained by acting with $\tilde{\alpha}_{-\frac{1}{2}}^{10} \tilde{\lambda}_{-\frac{1}{2}}^a$ can be made gauge invariant by acting with a fermion zero mode ψ_0^M or equivalently choosing the opposite chirality for these spinor states. So for massless

boundary fermion states, the effect of the GSO projection is to eliminate $\Psi_{\dot{\alpha}}^M, \psi_{\dot{\alpha}}^{[ab]}, \theta_{\dot{\alpha}}$ and $\Upsilon_{\dot{\alpha}}^a$, and to retain $\Psi_{\alpha}^M, \psi_{\alpha}^{[ab]}, \theta_{\alpha}$ and Υ_{α}^a .

The astute reader may recognize this pattern of representations for the gauge group $SO(33)$; it is the same pattern which occurs in the chiral fermion spectrum of type I string theory in ten dimensions, to which a single additional D9-brane and anti-D9-brane have been added [21].

The extra D9-brane enhances the gauge group to $SO(33)$, and the single $\overline{D9}$ -brane does not give rise to any gauge symmetry. In this theory there is a left-handed vector-spinor $\hat{\Psi}_{\alpha}^M$ coming from the closed string sector; a left-handed adjoint $\hat{\psi}_{\alpha}^{[ab]}$ coming from the D9- $\overline{D9}$ open strings; a right-handed $SO(33)$ vector $\hat{\Upsilon}_{\alpha}^a$, coming from the D9- $\overline{D9}$ open strings; and a left-handed singlet $\hat{\theta}_{\alpha}$ coming from the $\overline{D9}$ - $\overline{D9}$ open strings.

5.2. Generalization to higher n

The agreement between the gauge group and chiral fermion spectrum of the type I+D9+ $\overline{D9}$ and $HO^{+(n)}/$ theories persists for higher n , where the number of brane-antibrane pairs added to the type I theory is n , and the $HO^{+(n)}/$ theory has a codimension- n fixed locus of a Z_2 involution. In this subsection we describe the $HO^{+(n)}/$ orbifold for and its spectrum for general n . Our description is abbreviated, as it runs in parallel to the $n = 1$ case.

Starting with the $10 + n$ -dimensional $HO^{+(n)}/$ theory, which has continuous gauge group $SO(32 + n)$, we separate the coordinates into two groups, which we label X^M , $M = 0, \dots, 9$ and Y^s , $s = 1, \dots, n$. Our Z_2 orbifold action inverts all the coordinates Y^s and simultaneously acts with a minus sign on all $32 + n$ $\tilde{\lambda}^a$ -fields.

The calculation of the massless fermion spectrum in the $HO^{+(n)}/$ theory goes through as for $n = 1$:

Table 11: Chiral fermion spectrum of $\text{HO}^{+(n)}/$ at an \mathbb{R}^n/Z_2 orbifold singularity of stable type.

state	spacetime field	$SO(32+n)$ rep.	$SO(n)$ rep.	$SO(9,1)$ spinor chirality
$\tilde{\alpha}_{-1}^M k^N, \alpha\rangle_{g_1 g_2}$	Ψ_α^M	1	1	$+$ and $+(\text{spin } \frac{3}{2})$
$\tilde{\lambda}_{-\frac{1}{2}}^a \tilde{\lambda}_{-\frac{1}{2}}^b k^N, \alpha\rangle_{g_1 g_2}$	$\psi_\alpha^{[ab]}$	$\Lambda^2[\mathbf{32+n}]$	1	+
$\tilde{\alpha}_{-\frac{1}{2}}^s \tilde{\lambda}_{-\frac{1}{2}}^a k^N, \dot{\alpha}\rangle_{g_1 g_2}$	$\Upsilon_\alpha^{a s}$	32+n	n	-
$\tilde{\alpha}_{-\frac{1}{2}}^s \tilde{\alpha}_{-\frac{1}{2}}^t k^N, \alpha\rangle_{g_1 g_2}$	$\theta_\alpha^{(st)}$	1	$\text{Sym}^2[\mathbf{n}]$	+

For arbitrary n , the spectrum of massless fermions is the same as that of the type I theory with n extra D9- $\overline{\text{D9}}$ pairs added [21].

In the type I theory there are also tachyons $\hat{T}^{a|s}$ which transform in the bifundamental of $SO(32+n) \times SO(n)$. They are the lowest lying states in the NS sector of the D9- $\overline{\text{D9}}$ open strings. In the twisted sector of the $\text{HO}^{+(n)}/$ theory, the role of the open string tachyon is played by the normal derivative $\nabla_s T^a_{|Y^t=0} = \partial_s T^a_{|Y^t=0}$ of the bulk tachyon T^a at the orbifold fixed locus $Y^t = 0$. It is the normal derivative which participates in the most relevant coupling between the boundary fermions and the bulk tachyon:

$$\mathcal{L}_{\text{INT}}^{(9+1)} = v_1 (\partial_s T^b)_{|Y^u=0} \bar{\psi}^{[ab]} \Upsilon^{a|s} + v_2 (\partial_t T^a)_{|Y^u=0} \bar{\theta}^{(st)} \Upsilon^{a|s} \quad (5.9)$$

where v_1 and v_2 are Yukawa couplings which can be determined by a three-point computation in the tree-level $\text{HO}^{+(n)}/$ theory.

The same couplings occur in the type I+ n D9+ n $\overline{\text{D9}}$ system (with different coeffi-

cients), with the correspondence

$$\begin{aligned}
\psi^{[ab]} &\rightarrow \hat{\psi}^{[ab]} \\
\Upsilon^{s,a} &\rightarrow \hat{\Upsilon}^{s,a} \\
\theta^{(st)} &\rightarrow \hat{\theta}^{(st)} \\
\frac{\partial T^a}{\partial Y^s} \Big|_{Y^u=0} &\rightarrow \hat{T}^{a|s}
\end{aligned} \tag{5.10}$$

Hatted quantities refer to type I fields.

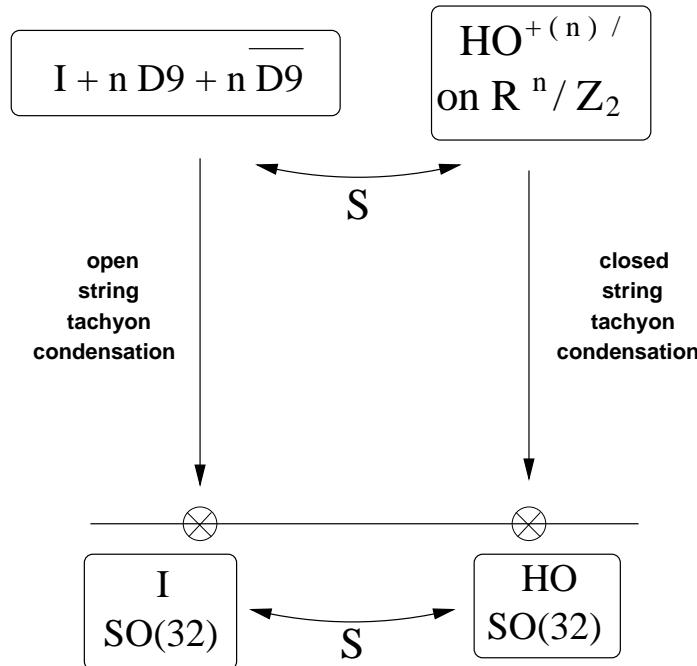


Fig. 3: An S-duality between critical open string theories and noncritical closed string theories. The \mathbb{R}^n/Z_2 singularity is of the stable, gauge-invariant type we have discussed, at which all tachyons have Dirichlet boundary conditions. The S-duality commutes with tachyon condensation. As we shall show in the next section, this diagram is a simplification of the true phase structure.

If the correspondence is to make sense as a true S-duality, the higher $SO(n)$ harmonics of the bulk fields must have some interpretation in terms of the ten-dimensional open string theory. The immediate challenge is that there are single-particle $SO(n)$ representations such as $T_{,s_1 s_2 \dots s_{2m+1}}^a$ which occur on the $HO^{+(n)}$ side, for which there is no corresponding $SO(n)$ representation among single-particle states on the type I side. The only possible resolution is that as the type I coupling is raised, the open string theory develops a tower

of $SO(n)$ non-singlet bound states, which fill out all the higher $SO(n)$ representations of spherical harmonics about the fixed point in the heterotic string theory. For instance, the heterotic field

$$\frac{\partial^{2m+1} T^a}{\partial Y^{s_1} \partial Y^{s_2} \dots \partial Y^{s_{2m+1}}} \Big|_{Y^u=0} \quad (5.11)$$

has the same $SO(n)$ quantum numbers as the composite type I field

$$\hat{T}^{a|s_1} \hat{T}^{b_1|s_2} \hat{T}^{b_1|s_3} \dots \hat{T}^{b_m|s_{2m}} \hat{T}^{b_m|s_{2m+1}} + (\text{permutations of } s_1, s_2, \dots, s_{2m+1}) \quad (5.12)$$

So the duality conjecture makes a surprising prediction about the behavior of the type I theory at strong coupling: it must enter a *nonabelian composite phase*, in which short-range forces bind open strings together into new single-particle states and resonances. These bound states are very different from the gauge-singlet glueball states which arise from long-range interactions in confining gauge theories. In ten dimensions, gauge interactions are always weak at long distances, so the binding forces leave $SO(n)$ color quantum numbers unconfined.

6. Compactification and symmetry breaking

6.1. Moduli spaces

The results of the previous section suggest an S-duality between type I string theory with n additional D9- $\overline{\text{D9}}$ pairs, and $\text{HO}^{+(n)}/$ theory in the presence of a certain kind of orbifold singularity. Such a duality would generalize the well-known duality between the supersymmetric backgrounds of the type I and HO string theories.

But we should be careful not to posit an equivalence between two theories which are manifestly inequivalent; the type I theory is a ten-dimensional string theory with ten dimensions' worth of momentum states for each mode of the string, a finite ten-dimensional Newton constant and gauge coupling, and so on.

Such a theory, at finite coupling, could not possibly provide an exact alternate description of a theory living in $10 + n$ compact dimensions.

Rather, we propose that at strong coupling, the type I string with n brane-antibrane pairs describes the type $\text{HO}^{+(n)}/$ string on a *family* of compact n -dimensional spaces, each of which has a single orbifold singularity of the type described in the previous section. The reason that the strong coupling physics of the type I string is described by a family of type $\text{HO}^{+(n)}/$ theories, rather than a single one, is that the $\text{HO}^{+(n)}/$ theory develops a

new branch of approximate moduli in the limit where the heterotic coupling is weak: the moduli of the CFT describing the compact n -dimensional space. If the space is a toroidal orbifold, for instance, such moduli are exactly massless at heterotic tree level, and can be lifted only by string loop effects.

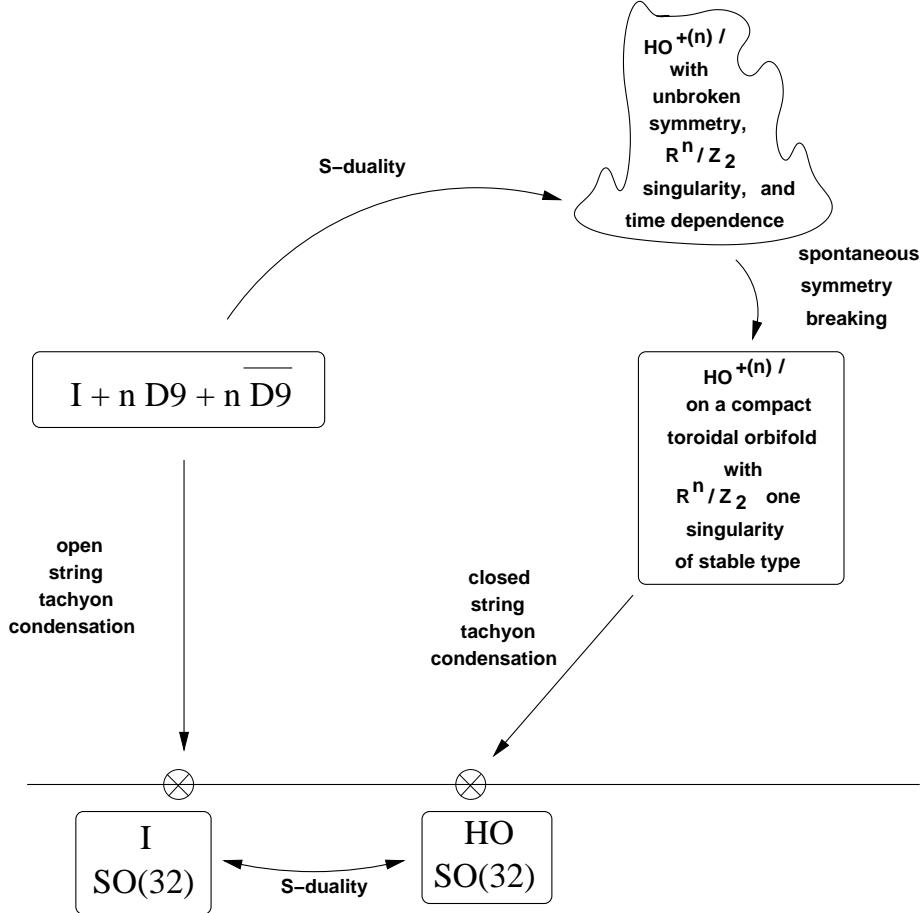


Fig. 4: A refined phase diagram illustrating the critical-supercritical S-duality. Raising the type I coupling leads to a potential which breaks some of the gauge symmetry spontaneously. The potential basin describes an approximate moduli space of toroidal orbifolds with one \mathbb{R}^n/Z_2 singularity of stable type.

In this section we shall construct compactifications of $\text{HO}^{+(n)}/$ theories for $n = 1, 2$, each with a single orbifold singularity of the stable type we have described in detail. Each will have another singularity with different boundary conditions which break some of the $SO(32 + n)$ gauge symmetry spontaneously. For $n = 2$ the compactification itself will break the $SO(n)$ gauge symmetry spontaneously as well. Nonetheless in both cases the spectrum of chiral fermions will be organized into multiplets of the $SO(n) \times SO(32 + n)$

gauge symmetry, and the fermions couple to the massive gauge bosons according to their gauge representation.

6.2. Symmetry-breaking boundaries for $\text{HO}^{+(1)}/$

Let us compactify the type $\text{HO}^{+(1)}/$ theory on an interval. And we would like to do this in such a way that the ten-dimensional tachyon potential has a supersymmetric global minimum describing a single universe in which the supersymmetric HO theory describes the degrees of freedom and their dynamics.

There is an obvious way to compactify the theory, by putting it on an interval with two boundaries of stable type. That compactification cannot describe a system with a ten-dimensional tachyon which condenses to a single supersymmetric universe. In order to have a single universe as the endpoint of tachyon condensation, it is important that our compact space have only one orbifold singularity of stable type.

If we were to have two singularities of stable type, then upon perturbation by a generic superpotential, the tachyon will condense to a state with two mutually disconnected regions. To illustrate this, we orbifold a circle $X^{10} \sim X^{10} + 2\pi R$ by an inversion $X^{10} \rightarrow -X^{10}, \psi^{10} \rightarrow -\psi^{10}$ combined with $\tilde{\lambda}^a \rightarrow -\tilde{\lambda}^a$. This leaves us with two boundaries of stable type. Now perturb this system by the superpotential

$$W = \tilde{\lambda}^{33} \sin \frac{X^{10}}{R} \quad (6.1)$$

The resulting potential $\sin^2 \frac{X^{10}}{R}$ has two zeroes, one at each fixed point. Each vacuum has unbroken worldsheet supersymmetry. The only massless worldsheet degrees of freedom are $\tilde{\lambda}^{1-32}$ and X^{0-9}, ψ^{0-9} , so each vacuum describes a copy of the worldsheet theory of the supersymmetric ten-dimensional HO string.

We can also verify that each vacuum contributes with the same sign to $\text{tr} [(-1)^{F_{RW}}]$ of the effective theory. To see this, remember that the right-moving fermion number of the standard HO theory is the one which inverts ψ^{0-9} but does not act on any of the $\tilde{\lambda}$, so this symmetry comes from $g_1 g_2$ in the microscopic theory on the worldsheet. Under this symmetry $\tilde{\lambda}^{33}$ and ψ^{10} are neutral, so despite the fact that their mass matrix changes sign as one goes between the two worldsheet vacua, the two vacua contribute with the same sign to $(-1)^{F_{RW}}$ in the effective theory. The endpoint of tachyon condensation is two stable, disconnected universes, each with its own graviton, $SO(32)$ gauge field, and chiral fermions.

To avoid such a situation, we would like to make the second wall of the interval an *unstable* type of boundary, with Neumann conditions for at least one of the tachyons. Let us first consider such a boundary in isolation.

In order to give a tachyon Neumann boundary conditions at a fixed locus, we need to let the corresponding fermion $\tilde{\lambda}^a$ be even, rather than odd, under the action that reverses X^{10}, ψ^{10} . In order to ensure modular invariance, we need the number of odd $\tilde{\lambda}$ to be one plus a multiple of sixteen. The simplest way to satisfy this requirement is to make only one of the $\tilde{\lambda}^a$, say $\tilde{\lambda}^{33}$, odd at the new wall. This choice breaks the $SO(33)$ continuous symmetry down to $SO(32)$.

6.3. An interval with one boundary of each type

There are two ways to study an interval with one boundary of each type. We could appeal directly to spacetime locality, and simply impose the appropriate boundary conditions on the spacetime fields at each boundary. This is the approach taken by the authors of [22]. We could also obtain the model with one boundary of each type by starting with the S^1/Z_2 model we described earlier, with two boundaries of stable type, and further orbifolding by g_3 , acting as follows ²

Table 12: The interval in type $HO^{+(1)}$ string theory with one stable and one unstable type of boundary can be thought of as an orbifold by an element g_3 of the interval with two boundaries of stable type. This table gives the charge assignments of worldsheet fields under g_3 .

object	g_3
X^{10}	$X^{10} \rightarrow \pi R - X^{10}$
ψ^{10}	-
$\tilde{\lambda}^{1-32}$	+
$\tilde{\lambda}^{33}$	-

² The conditions for level matching require that the momentum p_{10} be fractional in winding sectors such as $g_2 g_3$. The demonstration of level matching and closure of the OPE is straightforward but tedious; since winding strings do not play a role in our discussion, we omit the details. Related issues arise for the backgrounds discussed in [22].

Such an orbifold describes an interval of length $\frac{\pi R}{2}$ with two inequivalent boundaries, one at $X^{10} = 0$ with Dirichlet boundary conditions for all tachyons, and another at $\frac{\pi R}{2}$ with Dirichlet boundary conditions for T^{33} and Neumann boundary conditions for T^{1-32} .

Take R to be larger than string scale. In ten-dimensional terms, the most tachyonic modes have $k^M k_M + 2iV_M k^M = -\frac{2}{\alpha'} + \frac{1}{4R^2}$ and transform in the vector representation of the unbroken $SO(32)$. They come from acting on the state $\left| \sin \frac{x^{10}}{2R}, k^M, 0 \right\rangle_1$ with $\tilde{\lambda}_{-\frac{1}{2}}^{1-32}$. There is a thirty-third scalar mode which comes from acting on $\left| \sin \frac{x^{10}}{R}, k^M, 0 \right\rangle_1$ with $\tilde{\lambda}_{-\frac{1}{2}}^{33}$. This scalar satisfies $k^M k_M + 2iV_M k^M = -\frac{2}{\alpha'} + \frac{1}{R^2}$ and transforms along with the other thirty-two tachyons as the thirty-third component of the vector of the broken $SO(33)$.

What happens if we condense one of the tachyons which lives in the **32** of the unbroken $SO(32)$? The corresponding relevant superpotential perturbation is

$$W = \tilde{\lambda}^{1-32} : \sin \frac{X^{10}}{2R} : \quad (6.2)$$

For sufficiently large R we can just drop the normal ordering symbol and treat the perturbation as classical. The bosonic potential coming from this perturbation is

$$\sin^2 \frac{X^{10}}{2R} \quad (6.3)$$

This potential is nonvanishing everywhere except at the left-hand boundary. In the IR, the theory flows to a CFT describing strings in a single universe.

It is easy to see that the string theory in that universe is the supersymmetric HO theory in ten dimensions. The massless degrees of freedom in the infrared are the X^M and ψ^M , as well as $\tilde{\lambda}^{1-31}$ and $\tilde{\lambda}^{33}$.

We can also see that it is a stable, supersymmetric universe. Of the three Z_2 factors of the worldsheet gauge group, g_3 is spontaneously broken by the expectation value of x^{10} , and g_1, g_2 are unbroken. On the infrared degrees of freedom g_1 and g_2 act exactly as $(-1)^{F_{LW}}$ and $(-1)^{F_{RW}}$. So in the effective theory on the worldsheet there are two independent Z_2 gauge symmetries which act as the chiral Z_2 fermion number symmetry in the supersymmetric, ten dimensional type HO theory.

If instead of condensing one of the tachyons in the **32** we condense the thirty-third tachyon, we get a slightly different result. The state is

$$\tilde{\lambda}_{-\frac{1}{2}}^{33} \left| \sin \frac{x^{10}}{R}, k^M, 0 \right\rangle_1 \quad (6.4)$$

and the corresponding superpotential perturbation is

$$\tilde{\lambda}^{33} : \sin \frac{X^{10}}{R} : \quad (6.5)$$

This superpotential, and the corresponding bosonic potential, vanishes in two places, one at each fixed point.

The physics of the universe at $X^{10} = 0$ is still that of the supersymmetric HO theory, by the same set of arguments we have already given. The physics in the universe on the right is slightly different.

In the universe at $X^{10} = \frac{\pi R}{2}$, the unbroken symmetries are g_1 and g_3 . g_1 acts as $(-1)^{F_W} = (-1)^{F_{LW}} \cdot (-1)^{F_{RW}}$ on the infrared degrees of freedom, and g_3 acts trivially on all the infrared degrees of freedom. So the infrared CFT describes strings propagating in two disconnected universes; in the first universe the dynamics of string theory is governed by the critical, supersymmetric HO, and in the second universe the dynamics are governed by the critical, nonsupersymmetric HO'.

6.4. Spontaneous breaking of $SO(n)$: $HO^{+(2)}/$ on an $\mathbb{IR}\mathbb{P}_2$ orbifold.

In the example above, we compactified the theory in a way which gave rise to a single stable universe after tachyon condensation. Compactifying the theory in such a way involved breaking some of the continuous $SO(33)$ gauge symmetry spontaneously. We find that this is characteristic of the duality between unstable brane configurations and supercritical strings. The simple backgrounds we study *cannot* be understood simply as limits of the type I string which one can reach by changing the value of the dilaton only. At some point, one must shift the vacuum as well by giving an expectation value to some field charged under $SO(33)$. But if the compactification scale is large, as in the previous example, the spontaneous breaking can be viewed as small, in the sense that the masses of the gauge bosons are small compared to string scale.

Now we shall study an example with $n = 2$ and find that the $SO(2)$ gauge symmetry gets broken as well as well; to get a background which is static at tree level (modulo the variation of the dilaton) and has only ten noncompact dimensions, we must break some of the $SO(n)$ gauge symmetry spontaneously. But we will see that the compactification still has identifiable massive vector bosons of the Higgsed $SO(2)$ which couple to charged fermions as a gauge boson should.

We begin with $HO^{+(2)}/$ theory with the dimensions $X^{10,11}$ compactified on a T^2 with gauge bundle V . We define V by letting $\tilde{\lambda}^{33,34}$ be antiperiodic around both the X^{12}

direction and the X^{11} direction. Such a bundle can be obtained as a freely acting orbifold of a torus of twice the linear size.³ For simplicity, let the radii $R_{10} = R = R_{11}$ of the torus be equal.

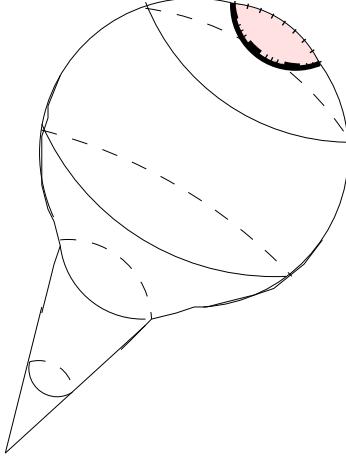


Fig. 5: A space with the topology of \mathbb{RP}_2 and a $U(1)$ isometry, with nonzero Ricci curvature and one \mathbb{R}^2/Z_2 orbifold singularity of stable type. Both the geometric $SO(2)$ and the current algebra $SO(34)$ are unbroken by this background. There are metric and dilaton tadpoles which lead to spontaneous gauge symmetry breaking. The pink disc at the top is a crosscap. This is the kind of time-dependent $HO^{+(2)}$ background one might obtain by taking the type I coupling to be large in the presence of two D9-branes and two $\overline{D9}$ -branes. We propose that it can relax to a toroidal orbifold background which is static except for the timelike linear dilaton.

Next, we orbifold this space by an action g_2 which just inverts both circle directions simultaneously; g_2 also acts with a minus sign on all thirty-four $\tilde{\lambda}^a$. This orbifold has the topology of a sphere and the geometry of a tetrahedron [23]. It has four fixed points, two of stable type and two unstable ones. By 'stable type', we mean what we meant earlier, that all tachyons at one of the fixed point of stable type $(X^{10}, X^{11}) = (0, 0)$ and $(\pi R, \pi R)$ have Dirichlet boundary conditions. At the two unstable fixed points,

$$(X^{10}, X^{11}) = (0, \pi R) \text{ and } (\pi R, 0) \quad (6.6)$$

the tachyons $T^{33,34}$ have Neumann boundary conditions and T^{1-32} have Dirichlet boundary conditions.

³ Again, modular invariance requires fractional KK momentum in winding sectors. See the previous footnote.

Finally we orbifold by another operation g_3 which exchanges the two fixed points of each type:

$$g_3 : (X^{10}, X^{11}) \mapsto (X^{10} + \pi R, \pi R - X^{11}) \quad (6.7)$$

This operation is orientation-reversing and so we let it act on the fermions as

$$(\tilde{\lambda}^{33}, \tilde{\lambda}^{34}) \mapsto (\tilde{\lambda}^{33}, -\tilde{\lambda}^{34}) \quad (6.8)$$

Neither g_3 nor g_2g_3 has fixed points on the torus; equivalently, g_3 acts freely on the S^2 . The action is orientation-reversing and the quotient has the topology of an \mathbb{RP}_2 . Like the interval described earlier, this orbifold has one fixed point of stable type, at $(X^{10}, X^{11}) = (0, 0)$. It has $SO(34)$ and $SO(2)$ unbroken near the fixed point, and Dirichlet boundary conditions for all the tachyons.

6.5. Coupling between fermions and gauge field

When we say that $SO(2)$ is unbroken near the fixed point, this is more than a manner of speaking. All string modes can be organized into $SO(2)$ multiplets according to the behavior of their wavefunctions near the fixed point.

First we identify the higgsed $SO(2)$ gauge boson which couples to the chiral fermions and tachyons according to their charges of the corresponding fields in the open string theory. They are built from the left moving current

$$J \equiv J^{[10|11]} \equiv : \sin \frac{X^{10}}{R} : \partial_- X^{11} - : \sin \frac{X^{11}}{R} : \partial_- X^{10}. \quad (6.9)$$

We will work in the limit where R is much larger than string scale. In this limit the breaking of gauge symmetry is small at the fixed point, and the weight of J is $(1, 0) + o(\frac{\alpha'}{R^2})$. We can decompose the gauge field vertex operator as

$$\mathcal{V}_{(2)} \equiv \exp\{-\phi\} c \tilde{c} (e \cdot \psi) \exp\{ik_M^{(2)} X^M\} J. \quad (6.10)$$

The two fermion vertex operators are

$$\mathcal{V}_{(1,3)} \equiv \exp\{-\phi/2\} c \tilde{c} \Theta_{\alpha_{(1,3)}} \exp\{ik_M^{(1,3)} X^M\} \tau_{(1,3)}, \quad (6.11)$$

where the τ is the internal piece of the vertex operator, made out of excited twist fields and current algebra fermions. We evaluate the first five factors in the correlator with ease:

$$\begin{aligned}
\langle \exp\{-\phi/2\}(z_1) \exp\{-\phi\}(z_2) \exp\{-\phi/2\}(z_3) \rangle_{S^2} &= z_{12}^{-\frac{1}{2}} z_{13}^{-\frac{1}{4}} z_{23}^{-\frac{1}{2}} \\
\langle \tilde{c}(z_1) \tilde{c}(z_2) \tilde{c}(z_3) \rangle_{S^2} &= \bar{z}_{12} \bar{z}_{13} \bar{z}_{23} \\
\langle c(z_1) c(z_2) c(z_3) \rangle_{S^2} &= z_{12} z_{13} z_{23} \\
\langle \Theta_{\alpha_{(1)}}(z_1) e \cdot \psi(z_2) \Theta_{\alpha_{(3)}}(z_3) \rangle_{S^2} &= \frac{1}{\sqrt{2}} (e_\mu \Gamma^\mu \Gamma^0)_{\alpha_{(1)} \alpha_{(3)}} z_{12}^{-\frac{1}{2}} z_{13}^{-\frac{3}{4}} z_{23}^{-\frac{1}{2}} \\
\left\langle \prod_{i=1}^3 \exp\{ik_M^{(i)} X^M\} \right\rangle_{S^2} &= \mathcal{I}(k_{(1)} + k_{(2)} + k_{(3)}) \prod_{\substack{i,j=1 \\ i < j}}^3 |z_{ij}|^{\alpha' k_{(i)} \cdot k_{(j)}} \tag{6.12}
\end{aligned}$$

Here all the formulae are familiar [14] except for the fifth; the function \mathcal{I} represents a formal divergent integral

$$\mathcal{I}(k) \equiv \int d^{10}x \exp\{(ik_M - 2V_M)x^M\} \tag{6.13}$$

over the zero modes of the embedding coordinates. To interpret this integral, rewrite $k_{(1,2,3)}$ in terms of the wave vectors of the fields with *canonical* normalization. Giving the fields unit kinetic term involves rescaling them by $\exp\{\Phi\}$, which shifts the momentum by iV . So the wave vectors k' of the canonically normalized fields are related to the wave vectors k of the fields with conventional string theory normalization by

$$k_{(i)} = k'_{(i)} + iV \tag{6.14}$$

So the zero mode integral can be rewritten as

$$\mathcal{I} \left(3iV + \sum_{i=1}^3 k'_{(i)} \right) = \int d^{10}x \exp \left\{ i \left(\sum_{i=1}^3 k'_{(i)} \right) \cdot x \right\} \exp\{V_M x^M\}, \tag{6.15}$$

which is just the standard overlap integral for wavefunctions of three canonical fields, with the integrand multiplied by an exponentially growing coupling constant.

We express the correlator this way in order to emphasize that the divergence of the amplitude comes entirely from a zero mode integral, and does nothing more than encode the enhancement of the interaction vertex by a coupling which depends on time.

To evaluate the sixth factor of the amplitude, note that the current J is a primary operator of weight $(1, 0) + o\left(\frac{\alpha'}{R^2}\right)$. Taking $\tau_{(1)}$ to be the complex conjugate $\tau_{(3)}^*$ of $\tau_{(3)}$, the easiest way to evaluate the correlator in the $X^{10,11}$ theory is in the operator formalism.

The mode expansion of J in the twisted sector is

$$J_j = -\frac{i\sqrt{\alpha'}}{4R} \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{\tilde{\alpha}_{j-k}^{10} \tilde{\alpha}_k^{11} - \tilde{\alpha}_k^{10} \tilde{\alpha}_{j-k}^{11}}{k} + o\left(\frac{\alpha'}{R^2}\right). \quad (6.16)$$

The only relevant mode is J_0 and the only relevant terms in J_0 are the ones with $k = \pm \frac{1}{2}$. So we can write

$$J_0 = -\frac{i\sqrt{\alpha'}}{R} \left(\tilde{\alpha}_{-\frac{1}{2}}^{10} \tilde{\alpha}_{\frac{1}{2}}^{11} - \tilde{\alpha}_{\frac{1}{2}}^{11} \tilde{\alpha}_{-\frac{1}{2}}^{10} \right) + (\text{other oscillators}) + o\left(\frac{\alpha'}{R^2}\right) \quad (6.17)$$

Using the CFT formula

$$\begin{aligned} & \langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \mathcal{O}_3(z_3) \rangle_{S^2} = \\ & c_{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3} \cdot z_{12}^{h_3-h_1-h_2} z_{13}^{h_2-h_1-h_3} z_{23}^{h_1-h_2-h_3} \bar{z}_{12}^{\tilde{h}_3-\tilde{h}_1-\tilde{h}_2} \bar{z}_{13}^{\tilde{h}_2-\tilde{h}_1-\tilde{h}_3} \bar{z}_{23}^{\tilde{h}_1-\tilde{h}_2-\tilde{h}_3} \end{aligned} \quad (6.18)$$

relating correlators on the sphere to structure coefficients of the OPE, we can reduce the three-point function to the calculation of

$$c_{\tau_{(1)}, J, \tau_{(3)}} = \langle \langle \tau_{(1)} | J_0 | \tau_{(3)} \rangle = \langle \tau_{(3)} | J_0 | \tau_{(3)} \rangle \quad (6.19)$$

There are three sets of spacetime fermions, corresponding to the cases where $|\tau_{(3)}\rangle$ is equal to

$$\begin{aligned} & \left| \tau^{(\psi^{[ab]})} \right\rangle = i \tilde{\lambda}_{-\frac{1}{2}}^a \tilde{\lambda}_{-\frac{1}{2}}^b |0\rangle_{g_1 g_2} \\ & \left| \tau^{(\Upsilon^{a+})} \right\rangle = \frac{1}{\sqrt{2}} \tilde{\lambda}_{+\frac{1}{2}}^a \left(\tilde{\alpha}_{-\frac{1}{2}}^{10} + i \tilde{\alpha}_{-\frac{1}{2}}^{11} \right) |0\rangle_{g_1 g_2} \\ & \left| \tau^{(\theta^{++})} \right\rangle = \frac{1}{2\sqrt{2}} \left(\tilde{\alpha}_{-\frac{1}{2}}^{10} + i \tilde{\alpha}_{-\frac{1}{2}}^{11} \right)^2 |0\rangle_{g_1 g_2} \end{aligned} \quad (6.20)$$

All three states are normalized to unity. A straightforward calculation yields

$$\begin{aligned} & \langle \langle \tau^{(\psi^{[ab]})} | J_0 | \tau^{(\psi^{[cd]})} \rangle = o\left(\frac{\alpha'}{R^2}\right) \\ & \langle \langle \tau^{(\Upsilon^{a-})} | J_0 | \tau^{(\Upsilon^{b+})} \rangle = \frac{\sqrt{\alpha'}}{R} + o\left(\frac{\alpha'}{R^2}\right) \\ & \langle \langle \tau^{(\theta^{--})} | J_0 | \tau^{(\theta^{++})} \rangle = \frac{2\sqrt{\alpha'}}{R} + o\left(\frac{\alpha'}{R^2}\right) \end{aligned} \quad (6.21)$$

In all three cases, the leading $\frac{1}{R}$ piece is proportional to the $SO(2)$ charge of the fermion.

These three-point functions encode the tree-level action for gauge fields and fermions. So the tree-level effective fermion action in ten dimensions is of the form

$$\frac{i}{\kappa_{10}^2} \int d^{10}x \sqrt{|G_{(10)}|} \exp\{-2\Phi_{10}\} [\bar{\psi} \Gamma^M \nabla_M \psi + \bar{\Upsilon} \Gamma^M \nabla_M \Upsilon + \bar{\theta} \Gamma^M \nabla_M \theta + \bar{\Psi}_N \Gamma^M \nabla_M \Psi^N], \quad (6.22)$$

where the connection ∇ is covariant with respect to $SO(2)$ as well as $SO(34)$. The relationship between Φ_{10} and the twelve dimensional dilaton Φ has the usual dependence on the volume of the compactification. The minimal couplings to the $SO(2)$ gauge field occur with the same relative coefficients as in the type I theory with 2 D9+ $\overline{D9}$ pairs added [21]. This constitutes a piece of evidence in support of our S-duality proposal.

The $O\left(\frac{\alpha'}{R^2}\right)$ terms represent corrections to the amplitude due to the spontaneous breaking of $SO(2)$ symmetry. Since the $SO(n)$ comes from a geometric rotation of the supercritical directions, it is unsurprising that compactification should break this symmetry. However it is important to note that this symmetry does organize the fermion spectrum of the twisted sector into exactly degenerate multiplets, and that the corresponding gauge bosons couple appropriately to those fields according to their $SO(2)$ charges.

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Appendix A. Fermion ground states in HO^+

In this appendix we explain how a reality condition can be imposed on states in the Ramond sectors g_2 and $g_1 g_2$ in a way that is covariant under the symmetries and consistent with the GSO projection.

We do not do a separate analysis for the bulk fermions of $HO^{+/-}$; all formulae relevant for bulk fermion states in that theory can be obtained from the ones in this appendix by replacing $SO(n) \rightarrow SO(n+32)$. Bulk fermions in the $HO^{+/-}$ theory are massive, and so they come in pairs of the representations discussed here.

A.1. The $\text{HO}^{+(1)}$ theory

In the Ramond sectors of the $\text{HO}^{+(1)}$ theory the fermion zero modes satisfy the same Clifford algebra as gamma matrices with signature $(11, 1)$. The fact that the states need only be a representation of $SO(10, 1)$ and not the full $SO(11, 1)$ allows us to impose a reality condition on Weyl spinors.

Let $\tilde{\Gamma}^\mu$, $\mu = 0, \dots, 10$ be a set of gamma matrices satisfying

$$\{\tilde{\Gamma}^\mu, \tilde{\Gamma}^\nu\} = 2\eta^{\mu\nu}. \quad (\text{A.1})$$

Since eleven is equal to three mod 8 and the signature is Lorentzian, we can impose the condition that all the $\tilde{\Gamma}^\mu$ be real. Then let

$$\Gamma^\mu \equiv \tilde{\Gamma}^\mu \otimes \sigma^1 \quad (\text{A.2})$$

and

$$\gamma^1 \equiv 1 \otimes \sigma^3 \quad (\text{A.3})$$

These matrices satisfy the relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} \quad \{\Gamma^\mu, \gamma^1\} = 0 \quad \{\gamma^1, \gamma^1\} = 2 \quad (\text{A.4})$$

and so together they generate a Clifford algebra of $SO(11, 1)$. With the identification

$$\psi_0^\mu \leftrightarrow \frac{1}{\sqrt{2}}\Gamma^\mu \quad \tilde{\chi}_0 \leftrightarrow \frac{1}{\sqrt{2}}\gamma^1, \quad (\text{A.5})$$

the gamma matrices provide a representation of the canonical anticommutation relations of the fermion zero modes.

The GSO projection restricts our choice of physical states to the subspace

$$64 i \left[\prod_\mu \psi_0^\mu \right] \cdot \tilde{\chi}_0 \leftrightarrow i \left[\prod_\mu \Gamma^\mu \right] \gamma^1 \equiv \hat{\Gamma} = \pm 1, \quad (\text{A.6})$$

where the sign \pm depends on the overall sign of the GSO projection, as well as on the sign contributed by any nonzero-frequency oscillators of the ψ and χ fields with which we act on the ground states.

In the basis we have chosen,

$$\hat{\Gamma} = 1 \otimes \sigma^2 \quad (\text{A.7})$$

Eigenspinors of this operator cannot be real. They are of the form

$$\Psi_{\pm} \equiv \begin{pmatrix} \psi_{\alpha} \\ \pm i \psi_{\alpha} \end{pmatrix}, \quad (\text{A.8})$$

where ψ_{α} is a representation of the $\tilde{\Gamma}^{\mu}$ and with the sign \pm depending on the eigenvalue \pm . The matrix $1 \otimes \sigma^1$ permutes the two entries in the column vector.

Independent of the basis, we know that Weyl spinors of $SO(8k+3, 1)$ cannot satisfy a covariant Majorana condition. However we only need a Majorana condition which is covariant under $SO(8k+2, 1)$, not the full $SO(8k+3, 1)$.

With that in mind, we impose

$$\Psi^* = \gamma^1 \Psi. \quad (\text{A.9})$$

In the basis we have chosen, that condition reduces to the reality of the spinor ψ_{α} . In fact our Majorana condition is actually invariant under

$$[O(10, 1) \times O(1)]_+ = O(10, 1), \quad (\text{A.10})$$

which includes reflections across odd numbers of planes, in addition to $SO(10, 1)$ transformations. Reflections across a hyperplane $x^{\mu} = 0$ are implemented by the operator $\Gamma^{\mu} \cdot \gamma^1$, not just Γ^{μ} . We use the notation $[O(10, 1) \times O(1)]_+$ to emphasize that the $\tilde{\chi}$ zero mode is also involved in the orientation-reversing transformations, and also because it makes the generalization to the higher $\text{HO}^{+(n)}$ theories simpler to see.

The end result for the Ramond sectors of the $\text{HO}^{+(1)}$ theory is that the states generated by the action of the fermion zero modes form a single Majorana spinor of $SO(10, 1)$ – in terms of spacetime field content, there are thirty-two real degrees of freedom before equations of motion are imposed.

A.2. Generalization to higher $\text{HO}^{+(n)}$, with $n \in 2\mathbb{Z}$

We break up the problem according to the value of n modulo eight.

For even n , we start with a set of matrices $\tilde{\Gamma}^{\mu}$ satisfying the Dirac algebra

$$\{\tilde{\Gamma}^{\mu}, \tilde{\Gamma}^{\nu}\} = 2\eta^{\mu\nu}. \quad (\text{A.11})$$

Let

$$\tilde{\Gamma} \equiv i^{\left(1 + \frac{(n+1)(n+2)}{2}\right)} \tilde{\Gamma}^0 \tilde{\Gamma}^1 \dots \tilde{\Gamma}^{n+9} \quad (\text{A.12})$$

be the Hermitean chirality matrix of $SO(n+9, 1)$. Similarly, also define a set $\tilde{\gamma}^A$ of gamma matrices of $SO(n)$, satisfying the Dirac algebra

$$\{\tilde{\gamma}^A, \tilde{\gamma}^B\} = 2\delta^{AB} \quad (\text{A.13})$$

and define

$$\tilde{\gamma} \equiv i^{\left(\frac{n(n-1)}{2}\right)} \tilde{\gamma}^1 \tilde{\gamma}^2 \dots \tilde{\gamma}^n \quad (\text{A.14})$$

to be the Hermitean matrix whose eigenvalue \pm distinguishes the two inequivalent spinor representations of $SO(n)$ for even n .

From these, we define the matrices

$$\begin{aligned} \Gamma^\mu &\equiv \tilde{\Gamma}^\mu \otimes 1 \\ \gamma^A &\equiv \tilde{\Gamma} \otimes \tilde{\gamma}^A \end{aligned} \quad (\text{A.15})$$

With the assignment

$$\psi_0^\mu \leftrightarrow \frac{1}{\sqrt{2}} \Gamma^\mu \quad \tilde{\chi}_0^A \leftrightarrow \frac{1}{\sqrt{2}} \gamma^A, \quad (\text{A.16})$$

the fermion zero modes satisfy the correct algebra

$$\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu} \quad \{\psi_0^\mu, \tilde{\chi}_0^A\} = 0 \quad \{\tilde{\chi}_0^A, \tilde{\chi}_0^B\} = \delta^{AB} \quad (\text{A.17})$$

For n even, the GSO projection correlates the eigenvalues of $\tilde{\Gamma}$ and $\tilde{\gamma}$ with one another:

$$\tilde{\Gamma}_{\alpha\beta} \tilde{\gamma}_{pq} |\beta, q, (\text{osc.})\rangle_{g_2} = \pm |\alpha, p, (\text{osc.})\rangle_{g_2}, \quad (\text{A.18})$$

where the sign \pm depends on the overall sign choice of the GSO projection in the Ramond sector, as well as on the sign contributed by the oscillators.

Next, we consider the reality condition.

Spinors of $SO(n+9, 1)$ and of $SO(n)$ have the same reality properties for all n . For $n \equiv 0 \pmod{8}$, the Weyl spinors of both $SO(n+9, 1)$ and $SO(n)$ are real; for $n \equiv 2$ and $6 \pmod{8}$, they are complex; and for $n = 4$ they are pseudo-real.

For $n \equiv 0 \pmod{8}$, we can simply make the matrices $\tilde{\Gamma}^\mu, \tilde{\gamma}^A$ all real, and impose the reality condition

$$\Psi^* = \Psi \quad (\text{A.19})$$

on states. In this case, the spacetime field content of a Ramond state (with fixed oscillator content) consists of two real spinors, say $\psi_{\alpha p} = \psi_{\alpha p}^*$ and $\psi_{\dot{\alpha}, \dot{p}} = \psi_{\dot{\alpha}, \dot{p}}^*$, where $\alpha, \dot{\alpha}$ run over bases of the positive and negative chirality representations of $SO(n+9, 1)$ and p, \dot{p} run over bases of the positive and negative chirality representations of $SO(n)$.

For $n \equiv 4 \pmod{8}$, we can do almost the same thing. Weyl spinors of $SO(n+9, 1)$ and $SO(9)$ are pseudoreal for these values of n , which means there is a conjugation matrix $B_{\alpha\beta}$ (respectively b_{pq}) which maps $SO(n+9, 1)$ spinors (respectively $SO(n)$ spinors) of definite chirality into their complex conjugates in a way which commutes with the continuous symmetry group and satisfies $B^*B = BB^* = -1$ (respectively $b^*b = bb^* = -1$). Further, this map is chirality preserving, i.e.

$$\tilde{\Gamma}B = B\tilde{\Gamma}^* \quad \tilde{\gamma}b = b\tilde{\gamma}^* \quad (\text{A.20})$$

Therefore if Ψ is a spinor of the Clifford algebra generated by Γ^μ and γ^A which satisfies

$$\Gamma\gamma\Psi = (\tilde{\Gamma} \otimes \tilde{\gamma})\Psi = \Psi, \quad (\text{A.21})$$

it is consistent to impose the reality condition

$$\Psi = C\Psi^*, \quad (\text{A.22})$$

with

$$C \equiv B \otimes b. \quad (\text{A.23})$$

To understand the meaning of this condition, we choose a basis in which $\Gamma \equiv \tilde{\Gamma} \otimes 1$ and $\gamma \equiv 1 \otimes \tilde{\gamma}$ are block diagonal:

$$\Gamma = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \gamma = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & +1 & \\ & & & -1 \end{pmatrix} \quad (\text{A.24})$$

Writing Ψ in components as

$$\begin{pmatrix} \psi_{\alpha p} \\ \psi_{\dot{\alpha} p} \\ \psi_{\alpha \dot{p}} \\ \psi_{\dot{\alpha} \dot{p}} \end{pmatrix}, \quad (\text{A.25})$$

the GSO constraint $\Gamma\gamma = \tilde{\Gamma} \otimes \tilde{\gamma} = 1$ says

$$\psi_{\dot{\alpha} p} = \psi_{\alpha \dot{p}} = 0 \quad (\text{A.26})$$

and the reality constraint imposes independent conditions on the two remaining components:

$$\psi_{\alpha p} = B_{\alpha\beta} b_{pq} \psi_{\beta q}^*, \quad \psi_{\dot{\alpha}\dot{p}} = B_{\dot{\alpha}\dot{\beta}} b_{\dot{p}\dot{q}} \psi_{\dot{\beta}\dot{q}}^* \quad (\text{A.27})$$

In terms of spacetime components, this leaves us again with two spinors $\psi_{\alpha p}$ and $\psi_{\dot{\alpha}\dot{p}}$, each of which satisfies its own reality condition.

For $n \equiv 2, 6 \pmod{8}$ the Weyl spinors of $SO(n+9, 1)$ and $SO(n)$ are neither real nor pseudoreal. For $SO(n+9, 1)$ there is a charge conjugation matrix B with the property that $\tilde{\Gamma}^\mu B = B \tilde{\Gamma}^{\mu*}$. By itself this means that a Dirac spinor of $SO(n+9, 1)$ is real or pseudoreal. But the operation

$$\psi \rightarrow B\psi^* \quad (\text{A.28})$$

is chirality-changing; if ψ has chirality \pm then $B\psi^*$ has chirality \mp . This is expressed by the equation $\tilde{\Gamma}B = -B\tilde{\Gamma}^*$.

The $SO(n)$ Clifford algebras for these n also have the property that there exists a matrix b with $\tilde{\gamma}^\mu b = b\tilde{\gamma}^{\mu*}$, and $\tilde{\gamma}b = -b\tilde{\gamma}^*$.

Therefore there is a natural complex-conjugation matrix for the $SO(n+9, 1) \times SO(n)$ Clifford algebra, and it is

$$C = B \otimes b \quad (\text{A.29})$$

and it has the property that

$$(\tilde{\Gamma} \otimes \tilde{\gamma})C = +C(\tilde{\Gamma} \otimes \tilde{\gamma})^* \quad (\text{A.30})$$

and

$$C^*C = CC^* = +1. \quad (\text{A.31})$$

So it is consistent to impose the condition

$$\Psi = C\Psi^* \quad (\text{A.32})$$

which in terms of components means

$$\psi_{\alpha p} = B_{\alpha\dot{\beta}} b_{p\dot{q}} \psi_{\dot{\beta}\dot{q}}^*. \quad (\text{A.33})$$

The chirality-changing nature of B and p means that the $B_{\alpha\beta}$, $B_{\dot{\alpha}\dot{\beta}}$, b_{pq} , and $b_{\dot{p}\dot{q}}$ components of B and b are all zero. In terms of spacetime field content, we need never refer to the spinor $\psi_{\dot{\alpha}\dot{p}}$, since we can eliminate it in terms of $\psi_{\alpha p}$. So for $n \equiv 2, 6 \pmod{8}$ we have a single complex Weyl spinor of $SO(n+9, 1)$ which also has definite chirality as a spinor representation of $SO(n)$.

A.3. Generalization to $\text{HO}^{+(n)}$, with $n \in 2\mathbb{Z} + 1$

We construct our gamma matrices as follows. Let

$$\begin{aligned}\Gamma^\mu &\equiv \tilde{\Gamma}^\mu \otimes 1 \otimes \sigma^1 \\ \gamma^A &\equiv 1 \otimes \tilde{\gamma}^A \otimes \sigma^3,\end{aligned}\tag{A.34}$$

where $\tilde{\Gamma}^\mu$ satisfy the Dirac algebra of $SO(n+9, 1)$ and $\tilde{\gamma}^A$ satisfy the Dirac algebra of $SO(n)$. The GSO projection restricts us to the subspace

$$1 \otimes 1 \otimes \sigma^2 = \pm 1,\tag{A.35}$$

for some sign \pm depending on the oscillator content of the state and the overall sign of the GSO projection. The solutions to the GSO constraint are spinors of the form

$$\Psi = \begin{pmatrix} \psi_{\alpha p} \\ \pm i\psi_{\alpha p} \end{pmatrix}\tag{A.36}$$

For $n \equiv 1, 7 \pmod{8}$, the reality condition generalizes straightforwardly from $n = 1$; in these dimensions we can choose all the gamma matrices to be real and imaginary, respectively. Then we can impose

$$\Psi = C\Psi^*,\tag{A.37}$$

with

$$C \equiv 1 \otimes 1 \otimes \sigma^3,\tag{A.38}$$

which is a covariant condition with respect to $SO(n+9, 1) \times SO(n)$. All pairwise products of Γ, γ respect the reality condition since they are both real and block diagonal in the last tensor factor.

For $n \equiv 3, 5 \pmod{8}$, the Dirac spinors of $SO(n+9, 1)$ and $SO(n)$ are pseudoreal, not real. So there is a conjugation matrix $B_{\alpha\beta}$ (respectively b_{pq}) which maps $SO(n+9, 1)$ spinors (respectively $SO(n)$ spinors) into their complex conjugates in a way which commutes with the symmetry group and satisfies $B^*B = BB^* = -1$ (respectively $b^*b = bb^* = -1$).

So we find that it is consistent to impose the condition

$$\Psi = C\Psi^*,\tag{A.39}$$

with

$$C \equiv B \otimes b \otimes \sigma^3.\tag{A.40}$$

In terms of the spinor $\psi_{\alpha p}$ this condition means that

$$\psi_{\alpha p} = B_{\alpha\beta} b_{pq} \psi_{\beta q}^* \quad (\text{A.41})$$

A.4. Summary

We summarize the results of this appendix in a table:

$n \pmod{8}$	reality of $\tilde{\Gamma}^\mu, \tilde{\gamma}^A$	B^*B and b^*b	$C_{SO(9,1) \times SO(n)}$	field content
0 ($n \neq 0$)	real	+1	$1 \otimes 1$	$\psi_{\alpha p} = \psi_{\alpha p}^*$ and $\psi_{\dot{\alpha}\dot{p}} = \psi_{\dot{\alpha}\dot{p}}^*$
1 ($n \neq 1$)	real	+1	$1 \otimes 1 \otimes \sigma^3$	$\psi_{\alpha p} = \psi_{\alpha p}^*$
2	-	both ± 1	$B \otimes b$	$\psi_{\alpha p}$
3	-	-1	$B \otimes b \otimes \sigma^3$	$\psi_{\alpha p} = B_{\alpha\beta} b_{pq} \psi_{\beta q}^*$
4	-	-1	$B \otimes b$	$\psi_{\alpha p} = B_{\alpha\beta} b_{pq} \psi_{\alpha p}^*$ and $\psi_{\dot{\alpha}\dot{p}} = B_{\dot{\alpha}\dot{\beta}} b_{\dot{p}\dot{q}} \psi_{\dot{\beta}\dot{q}}^*$
5	-	-1	$B \otimes b \otimes \sigma^3$	$\psi_{\alpha p} = B_{\alpha\beta} b_{pq} \psi_{\beta q}^*$
6	-	both ± 1	$B \otimes b$	$\psi_{\alpha p}$
7	imaginary	+1	$1 \otimes 1 \otimes \sigma^3$	$\psi_{\alpha p} = \psi_{\alpha p}^*$

Notes:

- For $n \equiv 2 \pmod{8}$ the gamma matrices $\tilde{\Gamma}^\mu$ and $\tilde{\gamma}^A$ can be made real, but this does not simplify the reality condition; in this basis $\tilde{\Gamma}$ is imaginary and off-diagonal, and so the γ^A are imaginary and off-diagonal as well.
- For even n , all spinors $\psi_{\alpha p}$ in the table should be taken to have definite chirality with respect to both $SO(n+9,1)$ and $SO(n)$.
- The formula for $n \equiv 1 \pmod{8}$ holds for $n = 1$ only in the sense that we understand the middle tensor factor to be a 1×1 dimensional matrix, equal to 1.

- For $n = 0$ there is only one chirality of spinor present, $\psi_{\alpha p} \equiv \psi_\alpha$. There is no $\psi_{\dot{\alpha}}$.
- The symmetry group which acts on spinors in $\text{HO}^{+(n)}$ is a double cover of $[O(9+n, 1) \times O(n)]_+$. The double cover is an index-two subgroup of a product of Pin groups: $[Pin(n+9, 1) \times Pin(n)]_+$, which is strictly larger than $Spin(9+n, 1) \times Spin(n)$. For more properties of the Pin groups see for example [24].
 - The additional elements are generated by reflections across a hyperplane $x^1 = 0$ combined with reflection on one gauge index, say $p = 1$. The action of this element on spinors is $\Psi \rightarrow \Gamma^1 \gamma^1 \Psi$.
 - For each n , the total number of real spinor components is 2^{n+4} . The fermion ground states form a *single* irreducible representation of $[Pin(n+9, 1) \times Pin(n)]_+$.

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